Starobinsky Inflation and Dark Energy and Dark Matter Effects from Quasicrystal Like Spacetime Structures

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Abstract

The goal of this work on mathematical cosmology and geometric methods in modified gravity theories, MGTs, is to investigate Starobinsky-like inflation scenarios determined by gravitational and scalar field configurations mimicking quasicrystal, QC, like structures. Such spacetime aperiodic QCs are different from those discovered and studied in solid state physics but described by similar geometric methods. We prove that inhomogeneous and locally anisotropic gravitational and matter field effective QC mixed continuous and discrete "ether" can be modelled by exact cosmological solutions in MGTs and Einstein gravity. The coefficients of corresponding generic off-diagonal metrics and generalized connections depend (in general) on all spacetime coordinates via generating and integration functions and certain smooth and discrete parameters. Imposing additional nonholonomic constraints, prescribing symmetries for generating functions and solving the boundary conditions for integration functions and constants, we can model various nontrivial torsion QC structures or extract cosmological Levi–Civita configurations with diagonal metrics reproducing de Sitter (inflationary) like and other type homogeneous inflation and acceleration phases. Finally, we speculate how various dark energy and dark matter effects can be modelled by off-diagonal interactions and deformations of a nontrivial QC like gravitational vacuum structure and analogous scalar matter fields.

Keywords: Off–diagonal cosmological metrics; effective gravitational and scalar field aperiodic structures; Starobinsky-Like inflation; dark energy and dark matter as quasicrystal structures.

Contents

1 Introduction

The Plank data [1] for modern cosmology prove a remarkable consistency of Starobinsky R^2–inflation theory [2] and a series of classical works on inflational cosmology [3, 4, 5, 6]. For reviews of results and changing of modern cosmology paradigms with dark energy and dark matter, and on modified gravity theories, MGTs, we cite [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Various cosmological scenarios studied in the framework of MGTs involve certain inhomogeneous and local anisotropic vacuum and non–vacuum gravitational configurations determined by corresponding type (effective) inflation potentials. In order to investigate such theoretical models, there are applied advanced numeric, analytic and geometric techniques which allow us to find exact, parametric and approximate solutions for various classes of nonlinear systems of partial differential equations, PDEs, considered in mathematical gravity and cosmology. The main goal of this paper is to elaborate on geometric methods in acceleration cosmology physics and certain models with effective quasicrystal like gravitational and scalar field structures.

The anholonomic frame deformation method, AFDM, (see review and applications in four dimensional, 4-d, and extra dimension gravity in [19, 20, 21]) allows to construct in certain general forms various classes of (non) vacuum generic off–diagonal solutions in Einstein gravity and MGTs. Such solutions may describe cosmological observation data and explain and predict various types of gravitational and particle physics effects [22, 23, 24, 25]. Following this geometric method, we define such nonholonomic frame transforms and deformations of connections (all determined by the same metric structure) when the gravitational and matter field motion equations decouple in general forms. In result, we can integrate also in general forms certain systems of gravitational and cosmologically important PDEs when the solutions are determined by generating and integration functions depending (in principle) on all spacetime coordinates and various classes of integration parameters.

The AFDM is very different from all other former methods applied for constructing exact solutions, when certain ansatz with higher symmetries (spherical, cylindrical, certain Lie algebra ones ...) are used for metrics which can be diagonalized by frame/coordinate transforms. For such "simplified" ansatz, the motion gravitational and/or cosmological equations transform into nonlinear systems of ordinary differential equations, ODEs, which can be solved in exact form for some special cases. For instance, there are considered diagonal ansatz for metrics with dependence on one radial (or time like) coordinate and respective transforms of PDEs to ODEs in order to construct black hole (or homogeneous cosmological) solutions, when the physical effects are computed for integration constants determined by respective physical constants. The priority of the AFDM is that we can use various classes of generating and integration functions in order to construct exact and parametric solutions determined by nonholonomic

nonlinear constraints and transforms, with generic off-diagonal metrics, generalized connections and various effective gravitational and matter fields configurations. Having constructed a class of general solutions, we can always impose additional constraints at the end (for instance, zero torsion, necessary boundary and symmetry conditions, or to consider homogeneous and isotropic configurations with nontrivial topology etc.) and search for possible limits to well known physically important solutions. Here we note that attempting to find exact solutions for nonlinear systems we lose a lot of physically important classes of solutions if we make certain approximations with "simplified" ansatz from the very beginning. Applying the AFDM, we can generate certain very general families of exact/parametric solutions after which there are possibilities to consider additional nonholonomic (non integrable) constraints and necessary type symmetry/ topology / boundary / asymptotic conditions which allow us to explain and predict certain observational and/or experimental data.

Modern acceleration cosmology data [1] show on existence of certain complex net–work filament and aperiodic type structures (with various fractal like, diffusion processes, nonlinear wave interactions etc.) determined by dark energy of the Universe and distribution of dark matter, with "hidden" frame support for respective meta–galactic and galactic configurations. Such configurations can be modelled by numeric and/or analytic geometric methods as deformations of an effective quasicrystal, QC, structure for the gravitational and fundamental scalar fields. Similar ideas were proposed many years ago in connection both with inflationary cosmology and quasicrystal physics [26, 27, 28, 6]. On early works and modern approaches to QC mathematics and physics, we cite [29, 30, 31, 32, 33, 34, 35, 36, 37] and references therein. Here we note that it is not possible to introduce directly a QC like locally anisotropic and inhomogeneous structure described by solutions of (modified) Einstein equations if we restrict the geometric approach only for cosmological models based only on the Friedman–Lemaître-Robertson-Worker, FLRW, metric. The geometric objects and physical values with homogeneous and isotropic metrics are determined by solutions with integration constants and this does not allow to elaborate on realistic an physically motivated cosmological configurations with QC nontrivial vacuum and non–vacuum structures. Realistic QC like aperiodic structures can not be described only via integration constants or certain Bianchi / Killing type and Lie group symmetries with structure constants. Solutions with nontrivial gravitational vacuum structure and respective cosmological scenarios can be generated by prescribing via generating functions and generating sources possible observational QC configurations following the AFDM as in [22, 23, 24, 25]. In nonholonomic variables, we can describe formation and evolution of QG structures as generalized geometric flow effects, see partner work [38] as recent developments and applications in physics of R. Hamilton and G. Pereman's theory of Ricci flows [39, 40]. Here, we emphasize that Lyapunov type functionals (for free energy) are used both in QC and Ricci flow evolution theories. For geometric evolution theories, such generalize entropy type functionals are known as Perelman's functionals with associated thermodynamical variables. The purpose of a planned series of papers is to study generic off–diagonal cosmological solutions with aperiodic order for (modified) Ricci solitons. In explicit form, the main goal of this article is to provide a geometric proof that aperiodic QC structures of vacuum and nonvacuum solutions of gravitational and scalar matter field equations MGTs and GR result in cosmological solutions mimicking Starobinsky like inflations and dark energy and dark matter scenarios which are compatible both with the

accelerating cosmology paradigm and observational cosmological data. We shall follow also certain methods for the mathematics of aperiodic order structures summarized in [41].

This paper is structured as follows. Section 2 is devoted to geometric preliminaries and main formulas for generating generic off–diagonal cosmological solutions in MGTs and GR (for proofs and details, readers are directed to review papers [19, 22, 23, 24]). In section 3, we elaborate on methods of constructing exact solutions with gravitational and scalar field effective QC structure. We study how aperiodic structures can be defined by generating functions and matter field sources adapted to off-diagonal gravitational and matter field interactions and evolution processes. Then, in section 4, we prove that Starobinsky like inflation scenarios can be determined by effective QC gravitational and adapted scalar field configurations. We also outline in that section a reconstructing formalism for analogous QC cosmological structure, with dark energy and dark matter effects. In section 5, we study certain MGTs configurations when dark energy and dark matter physics is modelled by QCs gravitational and effective scalar field configurations. Finally, conclusions and discussion are provided in section 6.

2 Generating Aperiodic Cosmological Solutions in R^2 Gravity

Let us outline a geometric approach to constructing exact solutions, (for the purposes of this paper), with aperiodic continuous and/or discrete, in general, inhomogeneous and locally anisotropic cosmological structures in (modified) gravity theories. We consider a general off–diagonal cosmological metrics $\mathbf{g}$ on a four dimensional, 4d, pseudo–Riemannian manifold $\mathbf{V}$ which can be parameterized via certain frame transforms as a distinguished metric, d–metric, in the form

$$\begin{aligned} \mathbf{g} &= g_{\alpha\beta}(u)\mathbf{e}^{\alpha}\otimes\mathbf{e}^{\beta} = g_i(x^k)dx^i\otimes dx^i + g_a(x^k,y^b)\mathbf{e}^a\otimes\mathbf{e}^b \qquad (1)\\ &= g_{\alpha\beta}(u)\mathbf{e}^{\alpha}\otimes\mathbf{e}^{\beta}, g_{\alpha'\beta'}(u) = g_{\alpha\beta}\mathbf{e}^{\alpha}{}_{\alpha'}\mathbf{e}^{\beta}{}_{\beta'},\\ \mathbf{e}^a &= dy^a + N_i^a(u^{\gamma})dx^i \text{ and } \mathbf{e}^{\alpha} = \mathbf{e}^{\alpha}{}_{\alpha'}(u)du^{\alpha'}. \qquad (2) \end{aligned}$$

In these formulas, the local coordinates $u^{\gamma} = (x^k, y^c)$, or $u = (x, y)$, when indices run respective values $i, j, k, ... = 1, 2$ and $a, b, c, ... = 3, 4$ are for conventional 2+2 splitting on$\mathbf{V}$ of signature $(+++-)$, when $u^4 = y^4 = t$ is a time like coordinate and $u^{\grave{\imath}} = (x^i, y^3)$ are spacelike coordinates with $\grave{\imath}, \grave{\jmath}, \grave{k}, ... = 1, 2, 3$. The values $\mathbf{N} = \{N_i^a\} = N_i^a$ define a N–adapted decomposition of the tangent bundle $T\mathbf{V} = hT\mathbf{V} \oplus vT\mathbf{V}$ into conventional horizontal, h, and vertical, v, subspaces[1]. Such a geometric splitting is nonholonomic because the basis $\mathbf{e}^{\alpha} = (x^i, \mathbf{e}^a)$ is dual to $\mathbf{e}_{\alpha} = (\mathbf{e}_i, e_a)$

$$e_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, e_a = \partial_a = \partial/\partial y^a,$$

[1]If it will not be a contrary statement for an explicit formula, we shall use the Einstein rule on summation on "up-low" cross indices. Such a system of "N–adapted notations" with boldface symbols for a nontrivial nonlinear connection, N–connection structure determined by $\mathbf{N}$ and related nonholonomic differential geometry is explained in details in [19, 22, 23, 24, 25] and references therein. We omit such considerations in this work.

which is nonholonomic (equivalently, non-intergrable, or anholonomic) if the commutators

$$\mathbf{e}_{[\alpha}\mathbf{e}_{\beta]} := \mathbf{e}_\alpha\mathbf{e}_\beta - \mathbf{e}_\beta\mathbf{e}_\alpha = C^\gamma_{\alpha\beta}(u)\mathbf{e}_\gamma$$

contain nontrivial anholonomy coefficients $C^\gamma_{\alpha\beta} = \{C^b_{ia} = \partial_a N^b_i, C^a_{ji} = \mathbf{e}_j N^a_i - \mathbf{e}_i N^a_j\}$. If such coefficients are not trivial, a N–adapted metric (1) can not be diagonalized in a local finite, or infinite, spacetime region with respect to coordinate frames. Such metrics are generic off-diagonal and characterized by six independent nontrivial coefficients from a set $\mathbf{g} = \{g_{\alpha\beta}(u)\}$. A frame is holonomic if all corresponding anholonomy coefficients are zero (for instance, the coordinate frames).

On $\mathbf{V}$, we can consider a distinguished connection, d–connection, $\mathbf{D}$, structure as a metric–affine (linear) connection preserving the N–connection splitting under parallel transports, i.e. $\mathbf{D} = (hD, vD)$. We denote the torsion of $\mathbf{D}$ as $\mathcal{T} = \{\mathbf{T}^\alpha_{\beta\gamma}\}$, where the coefficients can be computed in standard form with respect to any (non) holonomic basis. For instance, the well known Levi–Civita, LC, connection ∇ is a linear connection but not a d–connection because it does not preserve under general frame/coordinate transforms a h-v–splitting. Prescribing any d–metric and N–connection structure, we can work on $\mathbf{V}$ in equivalent form with two different linear connections:

$$(\mathbf{g}, \mathbf{N}) \to \begin{cases} \nabla : & \nabla\mathbf{g} = 0;\ {}^\nabla\mathcal{T} = 0, \text{ for the LC–connection} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\mathbf{g} = 0;\ h\widehat{\mathcal{T}} = 0, v\widehat{\mathcal{T}} = 0, hv\widehat{\mathcal{T}} \neq 0, \text{ for the canonical d–connection .} \end{cases} \tag{3}$$

In this formula, the $\widehat{\mathbf{D}} = h\widehat{\mathbf{D}} + v\widehat{\mathbf{D}}$ is completely defined by $\mathbf{g}$ for any prescribed N–connection structure $\mathbf{N}$. There is a canonical distortion relation

$$\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}. \tag{4}$$

The distortion distinguished tensor, d-tensor, $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^\alpha{}_{\beta\gamma}[\widehat{\mathbf{T}}^\alpha{}_{\beta\gamma}]\}$, is an algebraic combination of the coefficients of the corresponding torsion d-tensor $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^\alpha{}_{\beta\gamma}\}$ of $\widehat{\mathbf{D}}$. All such values are completely defined by data $(\mathbf{g}, \mathbf{N})$ being adapted to the N–splitting. It should be noted that $\widehat{\mathcal{T}}$ is a nonholonomically induced torsion determined by $(C^\gamma_{\alpha\beta}, \partial_\beta N^a_i, g_{\alpha\beta})$. It is different from that considered, for instance, in the Einstein–Cartan, or string theory, where there are considered additional field equations for torsion fields. We can re–define all geometric constructions for $\widehat{\mathbf{D}}$ in holonomic or nonholonomic variables for ∇ when the torsion vanishes in result of nonholonomic deformations.[2]

[2]Using a standard geometric techniques, the torsions, $\widehat{\mathcal{T}}$ and ${}^\nabla\mathcal{T} = 0$, and curvatures, $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^\alpha{}_{\beta\gamma\delta}\}$ and ${}^\nabla\mathcal{R} = \{R^\alpha{}_{\beta\gamma\delta}\}$ (respectively, for $\widehat{\mathbf{D}}$ and ∇) are defined and can be computed in coordinate free and/or coefficient forms. We can define the corresponding Ricci tensors, $\widehat{\mathcal{R}}ic = \{\widehat{\mathbf{R}}_{\ \beta\gamma} := \widehat{\mathbf{R}}^\gamma{}_{\alpha\beta\gamma}\}$ and $Ric = \{R_{\ \beta\gamma} := R^\gamma{}_{\alpha\beta\gamma}\}$, when the Ricci d-tensor $\widehat{\mathcal{R}}ic$ is characterized h-v N-adapted coefficients, $\widehat{\mathbf{R}}_{\alpha\beta} = \{\widehat{R}_{ij} := \widehat{R}^k{}_{ijk},\ \widehat{R}_{ia} := -\widehat{R}^k{}_{ika},\ \widehat{R}_{ai} := \widehat{R}^b{}_{aib},\ \widehat{R}_{ab} := \widehat{R}^c{}_{abc}\}$. We can also define two different scalar curvature, $R := \mathbf{g}^{\alpha\beta}R_{\alpha\beta}$ and $\widehat{\mathbf{R}} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}$. Following the two connection approach (3), the (pseudo) Riemannian geometry can be equivalently described by two different geometric data $(\mathbf{g}, \nabla)$ and $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. Using the canonical distortion relation (4), we can compute respective distortions $\widehat{\mathcal{R}} = {}^\nabla\mathcal{R} + {}^\nabla\mathcal{Z}$ and $\widehat{\mathcal{R}}ic = Ric + \widehat{\mathcal{Z}}ic$ and ${}^\nabla\mathcal{Z}$ and $\widehat{\mathcal{Z}}ic$.

The action $\mathcal{S}$ for a quadratic gravity model with $\widehat{\mathbf{R}}^2$ and matter fields with Lagrange density ${}^m\mathcal{L}(\mathbf{g},\mathbf{N},\varphi)$ is postulated in the form

$$\mathcal{S} = M_P^2 \int d^4u \sqrt{|\mathbf{g}|}[\widehat{\mathbf{R}}^2 + {}^m\mathcal{L}], \tag{5}$$

where the Plank mass M_P is determined by the gravitational constant. For simplicity, we consider in this paper actions ${}^m\mathcal{S} = \int d^4u\sqrt{|\mathbf{g}|}\ {}^m\mathcal{L}$ depending only on the coefficients of a metric field and not on their derivatives. In N–adapted form, the energy–momentum d–tensor can be computed

$${}^m\mathbf{T}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^m\mathcal{L})}{\delta\mathbf{g}^{\alpha\beta}} = {}^m\mathcal{L}\mathbf{g}^{\alpha\beta} + 2\frac{\delta({}^m\mathcal{L})}{\delta\mathbf{g}_{\alpha\beta}}.$$

In next sections, we shall chose such dependencies of ${}^m\mathcal{L}$ on (effective) scalar fields φ which will allow to model cosmological scenarios with dark mater and dark energy in MGTs in a compatible form nontrivial quasicrystal like gravitational and matter fields. The action $\mathcal{S}$ (5) results in the field equations

$$\widehat{\mathbf{R}}_{\mu\nu} = \mathbf{\Upsilon}_{\mu\nu}, \tag{6}$$

where $\mathbf{\Upsilon}_{\mu\nu} = {}^m\mathbf{\Upsilon}_{\mu\nu} + \widehat{\mathbf{\Upsilon}}_{\mu\nu}$, for ${}^m\mathbf{\Upsilon}_{\alpha\beta} = \frac{1}{2M_P^2}\ {}^m\mathbf{T}_{\alpha\beta}$ and $\widehat{\mathbf{\Upsilon}}_{\mu\nu} = (\frac{1}{4}\widehat{\mathbf{R}} - \frac{\widehat{\Box}\ \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}})\mathbf{g}_{\mu\nu} + \frac{\widehat{\mathbf{D}}_\mu\widehat{\mathbf{D}}_\nu\ \widehat{\mathbf{R}}}{\widehat{\mathbf{R}}}$,

and $\widehat{\Box} := \widehat{\mathbf{D}}^2 = \mathbf{g}^{\mu\nu}\widehat{\mathbf{D}}_\mu\widehat{\mathbf{D}}_\nu$. For $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}\to 0} = \nabla$, the equations can be re-defined via conformal transforms $g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = g_{\mu\nu}e^{-\ln|1+2\widetilde{\varphi}|}$, for $\sqrt{2/3}\varphi = \ln|1+2\widetilde{\varphi}|$, which introduces a specific Lagrange density for matter into the gravitational equations with effective scalar fields. Such a construction was used in the Starobinsky modified cosmology model [2]. In N–adapted frames, such a scalar field density can be chosen

$${}^m\widehat{\mathcal{L}} = -\frac{1}{2}\mathbf{e}_\mu\varphi\ \mathbf{e}^\mu\varphi - {}^\varphi V(\varphi) \tag{7}$$

resulting in matter field equations

$$\widehat{\Box}\varphi + \frac{d\ {}^\varphi V(\varphi)}{d\varphi} = 0.$$

In the above formula, we consider a nonlinear potential for scalar field ϕ

$$\begin{aligned} {}^\varphi V(\varphi) &= \varsigma^2(1-e^{-\sqrt{2/3}\varphi})^2, \varsigma = const, \\ \text{when } {}^\varphi V(\varphi \gg 0) &\to \varsigma^2,\ {}^\varphi V(\varphi=0)=0,\ {}^\varphi V(\varphi \ll 0) \sim \varsigma^2 e^{-2\sqrt{2/3}\varphi}. \end{aligned} \tag{8}$$

To apply such geometric methods in GR and MGTs is motivated by the fact that various types of gravitational and matter field equations rewritten in nonholonomic variables $(\mathbf{g},\mathbf{N},\widehat{\mathbf{D}})$ can be decoupled and integrated in certain general forms following the AFDM. This is not possible if we work from the very beginning with the data $(\mathbf{g},\nabla)$. Nevertheless, necessary type LC-configurations can be extracted from certain classes of solutions of (modified) gravitational field equations if additional conditions resulting in zero values for the canonical d-torsion, $\widehat{\mathcal{T}} = 0$, are imposed (considering some limits $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}\to 0} = \nabla$).

The aim of this work, we shall study scalar fields potentials $V(\varphi)$ modified by effective quasicrystal structures, $\varphi \to \varphi = \varphi_0 + \psi$, where $\psi(x^i, y^3, t)$ with crystal, or QC, like phases described by periodic or quasi-periodic modulations. Such modifications can be modelled in dynamical phase field crystal, PFC, like form [42]. The corresponding 3-d nonrelativistic dynamics is determined by a Laplace like operator ${}^3\triangle = ({}^3\nabla)^2$, with left label 3. In N–adapted frames with 3+1 splitting the equations for a local minimum conserved dynamics,

$$\partial_t \psi = {}^3\triangle \left[\frac{\delta F[\psi]}{\delta \psi} \right],$$

with two lenghs scales $l_{\underline{i}} = 2\pi/k_{\underline{i}}$, for $\underline{i} = 1, 2$. Such local diffusion process is described by a free energy functional

$$F[\psi] = \int \sqrt{|\ {}^3g\ |} dx^1 dx^2 dy^3 [\frac{1}{2}\psi\{-\epsilon + \prod_{\underline{i}=1,2} (k_{\underline{i}}^2 + {}^3\triangle)^2\}\psi + \frac{1}{4}\psi^4],$$

where $|\ {}^3g\ |$ is the determinant of the 3-d space metric and ϵ is a constant. For simplicity, we restrict our constructions only for non-relativistic diffusion processes, see Refs. [43, 44] for relativistic and N–adapted generalizations.

We shall be able to integrate in explicit form the gravitational field equations (6) and a d–metric (5) for (effective) matter field configurations parameterized with respect to N–adapted frames in the form

$$\mathbf{\Upsilon}^{\mu}{}_{\nu} = \mathbf{e}^{\mu}{}_{\mu'} \mathbf{e}_{\nu}{}^{\nu'} \mathbf{\Upsilon}^{\mu'}{}_{\nu'} [\ {}^m\mathcal{L}(\varphi + \psi),\ \widehat{\mathbf{\Upsilon}}_{\mu\nu}] = diag[\ {}_h\Upsilon(x^i)\delta^i_j, \Upsilon(x^i, t)\delta^a_b], \tag{9}$$

for certain vielbein transforms $\mathbf{e}^{\mu}{}_{\mu'}(u^{\gamma})$ and their duals $\mathbf{e}_{\nu}{}^{\nu'}(u^{\gamma})$, when $\mathbf{e}^{\mu} = \mathbf{e}^{\mu}{}_{\mu'} du^{\mu'}$, and $\mathbf{\Upsilon}^{\mu'}{}_{\nu'} = {}^m\mathbf{\Upsilon}^{\mu'}{}_{\nu'} + \widehat{\mathbf{\Upsilon}}^{\mu'}{}_{\nu'}$. The values ${}_h\Upsilon(x^i)$ and $\Upsilon(x^i, t)$ will be considered as generating functions for (effective) matter sources imposing certain nonholonomic frame constraints on (effective) dynamics of matter fields.

The system of modified Einstein equations (6) with sources (9) can be integrated in general form by such an off–diagonal asatz (see details in Refs. [19, 22, 23, 24, 25]):

$$\begin{aligned}
g_i &= e^{\ \psi(x^k)} \text{ as a solution of } \psi^{\bullet\bullet} + \psi'' = 2\ {}_h\Upsilon; \qquad (10)\\
g_3 &= \omega^2(x^i, y^3, t) h_3(x^i, t) = -\frac{1}{4}\frac{\partial_t(\Psi^2)}{\Upsilon^2}\left(h_3^{[0]}(x^k) - \frac{1}{4}\int dt \frac{\partial_t(\Psi^2)}{\Upsilon}\right)^{-1};\\
g_4 &= \omega^2(x^i, y^3, t) h_4(x^i, t) = h_4^{[0]}(x^k) - \frac{1}{4}\int dt \frac{\partial_t(\Psi^2)}{\Upsilon};\\
N_k^3 &= n_k(x^i, t) = {}_1n_k(x^i) + {}_2n_k(x^i) \int dt \frac{(\partial_t \Psi)^2}{\Upsilon^2 \left| h_3^{[0]}(x^i) - \frac{1}{4}\int dt\ \partial_t(\Psi^2)/\Upsilon \right|^{5/2}};\\
N_i^4 &= w_i(x^k, t) = \partial_i\ \Psi / \ \partial_t \Psi;\\
\omega &= \omega[\Psi, \Upsilon] \text{ is any solution of the 1st order system } \mathbf{e}_k \omega = \partial_k \omega + n_k \partial_3 \omega + w_k \partial_t \omega = 0.
\end{aligned}$$

In these formulas, $\Psi(x^i,t)$ and $\omega(x^i,y^3,t)$ are generating functions; ${}_h\Upsilon(x^i)$ and $\Upsilon(x^i,t)$ are respective generating h- and v–sources; ${}_1n_k(x^i)$, ${}_2n_k(x^i)$ and $h_a^{[0]}(x^k)$ are integration functions. Such values can be defined in explicit form for certain symmetry / boundary / asymptotic conditions which have to be considered in order to describe certain observational cosmological data (see next sections). The coefficients (10) generate exact and/or parametric solutions for any nontrivial $\omega^2=|h_3|^{-1}$. As a particular case, we can chose $\omega^2=1$ which allows to construct generic off–diagonal solutions with Killing symmetry on ∂_3.

The quadratic elements for such general locally anisotropic and inhomogeneous cosmological solutions with nonholonomically induced torsion are parameterized in this form:

$$\begin{aligned} ds^2 &= g_{\alpha\beta}(x^k,y^3,t)du^\alpha du^\beta = e^{\ \psi}[(dx^1)^2+(dx^2)^2]+ \\ & \omega^2\ \{h_3[dy^3+({}_1n_k+{}_2n_k\int dt\frac{(\partial_t\Psi)^2}{\Upsilon^2|h_3|^{5/2}})dx^k]^2-\frac{1}{4h_3}\left[\frac{\partial_t\Psi}{\Upsilon}\right]^2\ [dt+\frac{\partial_i\Psi}{\partial_t\Psi}dx^i]^2\}. \end{aligned} \tag{11}$$

In principle, we can consider that h_3 and Υ are certain generating functions when $\Psi[h_3,B,\Upsilon]$ is computed for $\omega^2=1$ from $\partial_t(\Psi^2)=B(x^i,t)/\Upsilon$ as a solution of

$$\Upsilon\left(h_3^{[0]}(x^k)-\int dtB\right)h_3(x^i,t)=-B.$$

This equation is equivalent to the second equation (10) up to re-definition of the integration function $h_3^{[0]}(x^k)$. Various classes of exact solutions with nontrivial nonholonomically induced torsion can be constructed, for instance, choosing data (Ψ,Υ) for solitonic like functions and/or for various singular, or discrete like structures. Such generic off–diagonal metrics can encode nontrivial vacuum and non-vacuum configurations, fractional and diffusion processes, and describe structure formation for evolving universes, effects with polarization of gravitational and matter field interaction constants, modified gravity scenarios etc., see examples in Refs. [19, 22, 23, 24, 43, 44].

The class of metrics (11) defines exact solutions for the canonical d–connection $\widehat{\mathbf{D}}$ in $\widehat{\mathbf{R}}^2$ gravity with nonholonomically induced torsion and effective scalar field encoded into a gravitationally polarized vacuum. We can impose additional constraints on generating functions and sources in order to extract Levi–Civita configurations. This is possible for a special class of generating functions and sources when for $\Psi=\check{\Psi}(x^i,t),\partial_t(\partial_i\check{\Psi})=\partial_i(\partial_t\check{\Psi})$ and $\Upsilon(x^i,t)=\Upsilon[\check{\Psi}]$, or $\Upsilon=const$. For such LC–solutions, we find some functions $\check{A}(x^i,t)$ and $n(x^k)$ when

$$w_i=\check{w}_i=\partial_i\check{\Psi}/\partial_t\check{\Psi}=\partial_i\check{A}\text{ and }n_k=\check{n}_k=\partial_kn(x^i).$$

The corresponding quadratic line element can be written

$$ds^2=\ e^{\ \psi}[(dx^1)^2+(dx^2)^2]+\omega^2\{h_3[dy^3+(\partial_kn)dx^k]^2-\frac{1}{4h_3}\left[\frac{\partial_t\check{\Psi}}{\Upsilon}\right]^2\ [dt+(\partial_i\check{A})dx^i]^2\}. \tag{12}$$

Both classes of solutions (11) and/or (12) posses additional nonlinear symmetries which allow to re–define the generation function and generating source in a form determined by an effective

(in the v-subspace) gravitational constant. For certain special parameterizations of $(\check{\Psi}, \Upsilon)$ and other coefficients, we can reproduce Bianchi like universes, extract FLRW like metrics, or various inhomogeneous and locally anisotropic configurations in GR. Using generic off-diagonal gravitational and matter field interactions, we can mimic MGTs effects, or model fractional/ diffusion / crystal like structure formation. Finally, we note that, such metrics (12) can not be localized in finite or infinite space time region if there are nontrivial anholonomy coefficients $C^{\gamma}_{\alpha\beta}$.

3 Modified Gravity with Quasicrystal Like Structures

To introduce thermodynamical like characteristics for gravitational and scalar field we consider an additional 3+1 splitting when off–diagonal metric ansatz of type (1), (11) (12) can be re-written in the form

$$\begin{aligned} \mathbf{g} &= b_i(x^k)dx^i \otimes dx^i + b_3(x^k, y^3, t)\mathbf{e}^3 \otimes \mathbf{e}^3 - \check{N}^2(x^k, y^3, t)\mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= dy^3 + n_i(x^k, t)dx^i, \quad \mathbf{e}^4 = dt + w_i(x^k, t)dx^i. \end{aligned} \tag{13}$$

In such a case, the 4–d metric $\mathbf{g}$ is considered as an extension of a 3–d metric $b_{\grave{i}\grave{j}} = diag(b_{\grave{i}}) = (b_i, b_3)$ on a family of 3-d hypersurfaces $\widehat{\Xi}_t$ parameterized by t. We have

$$b_3 = g_3 = \omega^2 h_3 \text{ and } \check{N}^2(u) = -\omega^2 h_4 = -g_4, \tag{14}$$

defining a lapse function $\check{N}(u)$. For such a double 2+2 and 3+1 fibration, $\widehat{\mathbf{D}} = (\widehat{D}_i, \widehat{D}_a) = (\widehat{D}_{\grave{i}}, \widehat{D}_t)$ (in coordinate free form, we write $({}^q\widehat{D}, {}^t\widehat{D})$). Similar splitting can be performed for the LC-operator, $\nabla = (\nabla_i, \nabla_a) = (\nabla_{\grave{i}}, \nabla_t) = ({}^q\nabla, {}^t\nabla)$. For simplicity, we elaborate the constructions for solutions with Killing symmetry on ∂_3.

3.1 Generating functions with 3d quasicristal like structure

Gravitational QC like structures can be defined by generic off–diagonal exact solutions if we chose a generating function $\Psi = \Phi$ as a solution of an evolution equation with conserved dynamics of type

$$\frac{\partial \Phi}{\partial t} = {}^b\widehat{\Delta}\left[\frac{\delta F}{\delta \Phi}\right] = -{}^b\widehat{\Delta}(\Theta\Phi + Q\Phi^2 - \Phi^3), \tag{15}$$

where the canonically nonholonomically deformed Laplace operator ${}^b\widehat{\Delta} := ({}^b\widehat{D})^2 = q^{\grave{i}\grave{j}}\widehat{D}_{\grave{i}}\widehat{D}_{\grave{j}}$ as a distortion of ${}^b\Delta := ({}^b\nabla)^2$ can be defined on any $\widehat{\Xi}_t$. Such distortions of differential operators can be always computed using formulas (4). The functional F in the evolution equation (15) defines an effective free energy (it can be associate to a model of dark energy, DE)

$$F[\Phi] = \int \left[-\frac{1}{2}\Phi\Theta\Phi - \frac{Q}{3}\Phi^3 + \frac{1}{4}\Phi^4\right]\sqrt{b}dx^1dx^2\delta y^3, \tag{16}$$

where $b = \det|b_{\grave{i}\grave{j}}|, \delta y^3 = \mathbf{e}^3$ and the operator Θ and parameter Q will be defined below. Such a configuration stabilized nonlinearly by the cubic term when the second order resonant

interactions can be varied by setting the value of Q. The average value $\overline{\Phi}$ of the generating function Φ is conserved for any fixed t. This means that $\overline{\Phi}$ can be considered as an effective parameter of the system and that we can choose $\overline{\Phi}_{|t=t_0}=0$ since other values can be re–defined and accommodated by altering Θ and Q. Further evolution can be computed for any solution of type (11) and/or (12).

The effective free energy $F[\Phi]$ defines an analogous 3-d phase gravitational field crystal (APGFC) model that generates modulations with two length scales for off–diagonal cosmological configurations. This model consists a nonlinear PDE with conserved dynamics. It describes (in general, relativistic) time evolution of Φ over diffusive time scales. For instance, we can elaborate such a APGFC model in a form including resonant interactions that may occur in the case of icosahedral symmetry considered for standard quasicrystals in [29, 30]. In this work, such gravitational structures will be defined by redefining Φ into respective generating functions Ψ or $\check{\Psi}$. Let us explain respective geometric constructions with changing the generating data $(\Psi,\ \Upsilon)\leftrightarrow(\Phi,\widetilde{\Lambda}=const)$ following the conditions

$$\begin{aligned}\frac{\partial_t(\Psi^2)}{\Upsilon} &= \frac{\partial_t(\Phi^2)}{\widetilde{\Lambda}}, \text{ which is equivalent to} \\ \Phi^2 &= \widetilde{\Lambda}\int dt\Upsilon^{-1}\partial_t(\Psi^2) \text{ and/or } \Psi^2=\widetilde{\Lambda}^{-1}\int dt\Upsilon\partial_t(\Phi^2).\end{aligned} \tag{17}$$

In result, we can write respective v- and hv–coefficients in (10) in terms of Φ (re–defining the integration functions),

$$\begin{aligned}h_3(x^i,t) &= -\frac{1}{4}\frac{\partial_t(\Phi^2)}{\Upsilon\widetilde{\Lambda}}\left(h_3^{[0]}(x^k)-\frac{\Phi^2}{4\widetilde{\Lambda}}\right)^{-1}=\frac{1}{\Upsilon}\frac{\partial_t(\Phi^2)}{\Phi^2-h_3^{[0]}(x^k)}; h_4(x^i,t)=h_4^{[0]}(x^k)-\frac{\Phi^2}{4\widetilde{\Lambda}}; \\ n_k &= {}_1n_k+{}_2n_k\int dt\ \frac{h_4[\Phi]}{\mid h_3[\Phi]|^{3/2}} \text{ and } w_i=\frac{\partial_i\ \Psi}{\partial_t\Psi}=\frac{\partial_i\ \Psi^2}{\partial_t(\Psi^2)}=\frac{\partial_i\ \int dt\Upsilon\ \partial_t(\Phi^2)}{\Upsilon\partial_t(\Phi^2)}.\end{aligned} \tag{18}$$

The nonlinear symmetry (17) allows us to change generate such effective sources (9) which allow to generate QC structures in self-consistent form when

$$\Upsilon(x^k,t)\rightarrow\ \Lambda=\ {}^f\Lambda+\ {}^\varphi\Lambda, \tag{19}$$

with associated effective cosmological constants in MGT, ${}^f\Lambda$, and for the effective QC structure, ${}^\varphi\Lambda$. We can identify $\widetilde{\Lambda}$ with Λ, or any other value ${}^f\Lambda$, or ${}^\varphi\Lambda$ depending on the class of models with effective gauge interactions we consider in our work.

Let us explain how the formation and stability of gravitational configurations with icosahedral quasicrystalline structures can be studied using a dynamical phase field crystal model with evolution equations (15). Such a 3-d QC structure is stabilized by nonlinear interactions between density waves at two length scales [30]. Using a generating function Φ, we elaborate a 3-d effective phase field crystal model with two length scales as in so-called Lifshitz–Petrich model [45]. The density distribution of matter mimics a "solid" or a "liquid " on the microscopic length. The role of operator Θ to allow two wave marginal numbers and to introduce possible spatio temporal chaos is discussed in [46, 30]. The effect is similar at metagalactic

scales when Φ has a two parametric dependence with $k=1$ (the system is weakly stable) and $k=1/\tau$ (where, for instance, for $\tau=2\cos\frac{\pi}{5}=1.6180$ we obtain the golden ratio, when the system is weakly unstable).

Choosing a QC type form for Φ and determining the coefficients of d–metric in the form (18), we generate a QC like structure for generic off–diagonal gravitational field interactions. Such a structure is formed by some type ordered arrangements of galaxies (as "atoms") with very rough rotation and translation symmetries. A more realistic picture of the observational data for the Universe is for a non crystal structure with lack of the translational symmetry but yet with certain discrete observations. There is certain analogy of such configurations for quasiperiodic two and three dimensional space like configurations, for instance, in metallic alloys, or nanoparticles, [as a review, see [46, 30, 45] and references therein] and at meta-galactic scales when the nontrivial vacuum gravitational cosmological structure is generated as we consider in this section.

3.2 Effective scalar fields with quasicrystal like structure

Following our system of notations, we shall put a left label "q" to the symbols for geometric/ physical object in order to emphasize that they encode an aperiodic QC geometric structure and write, for instance, $\left({}^{q}\mathbf{g},\ {}^{q}\widehat{\mathbf{D}},\ {}^{q}\varphi \right)$. We shall omit left labels for continuous configurations and/or if that will simplify notations and do not result in ambiguities.

The quadratic gravity theory with action (5) is invariant (both for ∇ and $\widehat{\mathbf{D}}$) under global dilatation symmetry with a constant σ,

$$g_{\mu\nu}\rightarrow e^{-2\sigma}g_{\mu\nu},\varphi\rightarrow e^{2\sigma}\widetilde{\varphi}. \tag{20}$$

We can pas from the Jordan to the Einstein frame with a redefinition $\varphi=\sqrt{3/2}\ln|2\widetilde{\varphi}|$ and obtain

$$^{\Phi}\mathcal{S}=\int d^4u\sqrt{|g|}\left(\frac{1}{2}\widehat{\mathbf{R}}-\frac{1}{2}\mathbf{e}_{\mu}\varphi\ \mathbf{e}^{\mu}\varphi-2\Lambda\right), \tag{21}$$

where the scalar potential ${}^{\varphi}V(\varphi)$ in (8) is transformed into an effective cosmological constant term Λ using $(\Psi,\ \Upsilon)\leftrightarrow(\Phi,\widetilde{\Lambda})$ (17). Such an integration constant can be positive / negative / zero, respectively for de Sitter / anti de Sitter / flat space.

The corresponding field equations derived from ${}^{E}\mathcal{S}$ are

$$\widehat{\mathbf{R}}_{\mu\nu}-\mathbf{e}_{\mu}\varphi\ \mathbf{e}_{\nu}\varphi-2\Lambda\mathbf{g}_{\mu\nu} = 0, \tag{22}$$

$$\widehat{\mathbf{D}}^2\varphi = 0. \tag{23}$$

We obtain a theory with effective scalar field adapted to a nontrivial vacuum QC structure encoded into $\mathbf{g}_{\mu\nu},\mathbf{e}_{\mu}$ and $\widehat{\mathbf{D}}$ as generic off–diagonal cosmological solutions. At the end of this section, we consider three examples of such QC gravitational-scalar field configurations as aperiodic and mixed continuous and discrete solutions of the gravitational and matter field equations (22) and (23).

3.2.1 Scalar field N–adapted to gravitational quasicrystals

In order to generate integrable off–diagonal solutions, we consider certain special conditions for the effective scalar field φ when $\mathbf{e}_\alpha\varphi = {}^0\varphi_\alpha = const$ in N-adapted frames. For such configurations, $\widehat{\mathbf{D}}^2\varphi = 0$. We restrict our models to configurations of ϕ, which can be encoded into N–connection coefficients

$$\mathbf{e}_i\varphi = \partial_i\varphi - n_i\partial_3\varphi - w_i\partial_t\varphi = {}^0\varphi_i;\ \ \partial_3\varphi = {}^0\varphi_3;\ \ \partial_t\varphi = {}^0\varphi_4;\quad \text{for } {}^0\varphi_1 = {}^0\varphi_2 \text{ and } {}^0\varphi_3 = {}^0\varphi_4. \tag{24}$$

This way we encode the contribution of scalar field configurations into additional source

$${}^\varphi\widetilde{\Upsilon} = {}^\varphi\widetilde{\Lambda}_0 = const \ \text{ and } \ {}^\varphi\Upsilon = {}^\varphi\Lambda_0 = const$$

even the gravitational vacuum structure is a QC modeled by Φ as a solution of (15).

3.2.2 Scalar and rescaled QC generating functions

The scalar field equations (15) can be solved if $\varphi = Z\Phi$, for $Z = const \neq 0$. The conditions (24) with ${}^0\varphi_1 = {}^0\varphi_2 = {}^0\varphi_3 = 0$ and nontrivial $\widehat{\Gamma}^4_{44} = -\partial_t h_4/h_4$ transform into

$$\partial_t\varphi = -\,{}^b\widehat{\Delta}(\Theta\varphi + Q\varphi^2 - \varphi^3), \tag{25}$$

$$\partial_i\varphi - w_i\partial_t\varphi = \partial_i\varphi - \frac{\partial_i\Phi}{\partial_t\Phi}\partial_t\varphi \equiv 0,$$

$$\widehat{\mathbf{D}}^2\varphi = h_4^{-1}(1 + \widehat{\Gamma}^4_{44})\partial_t\varphi = 0. \tag{26}$$

For $h_4(x^k, t)$ given by (18), we obtain nontrivial solutions of (26) if $1+\widehat{\Gamma}^4_{44} = 0$. This constraints additionally Φ, i.e. $\varphi = Z\Phi$, to the condition $2\partial_t\varphi = \frac{4h_4^{[0]}(x^k)}{\widetilde{\Lambda}Z^2\varphi} - \varphi$. Together with (25) we obtain that N–adapted scalar fields mimic a QC structure if

$$\widetilde{\Lambda}Z^2\varphi\left[\varphi - {}^b\widehat{\Delta}(\Theta\varphi + Q\varphi^2 - \varphi^3)\right] = 2h_4^{[0]}(x^k).$$

Using different scales, we can consider the energy of such QC scalar structures as hidden energies for dark matter, DM, modeled by φ, determined by an effective functional

$${}^{DM}F[\varphi] = \int\left[-\frac{1}{2}\varphi\widehat{\Theta}\varphi - \frac{\widehat{Q}}{3}\varphi^3 + \frac{1}{4}\varphi^4\right]\sqrt{b}dx^1dx^2\delta y^3, \tag{27}$$

where operators $\widehat{\Theta}$ and $\widehat{Q}$ have to be chosen in some forms compatible to observational data for the standard matter interacting with the DM. Even the QC structures for the gravitational fields (with QC configurations for the dark energy, DE) and for the DM can be different, we parameterize F and ${}^{DM}F$ in similar forms because such values are described effectively as exact solutions of Starobisky like model with quadratic Ricci scalar term. Here we note that such a similar φ–model was studied with a similar Lyapunov functional (effective free energy) ${}^{DM}F[\varphi]$ resulting in the Swift–Hohenberg equation (25), see details in Refs. [47, 45].

4 Aperiodic QC Starobinsky Like Inflation

The Starobinsky model described an inflationary de Sitter cosmological solution by postulating a quadratic on Ricci scalar action [2]. In nonholonomic variables, such MGTs were developed in [22, 23, 24, 25].

4.1 Inflation parameters determined by QC like structures

Although the Starobinsky cosmological model might appear not to involve any quasicrystal structure as we described in previous section, it is in fact conformally equivalent to a nonholonomic deformation of the Einstein gravity coupled to an effective QC structure that may drive inflation and acceleration scenarios. This follows from the fact that we can linearize the $\widehat{\mathbf{R}}^2$–term in (5) as we considered for the action (21). Let us introduce an auxiliary Lagrange field $\lambda(u)$ for a constant $\varsigma = 8\pi/3\mathcal{M}^2$ for a constant $\mathcal{M}$ of mass dimension one, with $\kappa^2 = 8\pi G$ for the Newton's gravitational constant $G = 1/M_P^2$ and Plank's mass, and perform respective conformal transforms with dilaton symmetry (20). We obtain that the action for our MGT can be written in three equivalent forms,

$$
\begin{aligned}
\mathcal{S} &= \frac{1}{2\kappa^2}\int d^4u\sqrt{|\mathbf{g}|}\left\{\widehat{\mathbf{R}}[\mathbf{g}]+\varsigma\widehat{\mathbf{R}}^2[\mathbf{g}]\right\}, \text{ with } \left\{\begin{array}{c}\mathbf{g}_{\mu\nu}\to\widetilde{\mathbf{g}}_{\mu\nu}=[1+2\varsigma\lambda(u)]\mathbf{g}_{\mu\nu}\\ \lambda(u)\to\varphi(u):=\sqrt{3/2}\ln[1+2\varsigma\lambda(u)]\end{array}\right., \\
&\rightarrowtail \frac{1}{2\kappa^2}\int d^4u\sqrt{|\mathbf{g}|}\left\{[1+2\varsigma\lambda(u)]\widehat{\mathbf{R}}[\mathbf{g}]-\varsigma\lambda^2(u)\right\} \\
&\rightarrowtail \frac{1}{2\kappa^2}\int d^4u\sqrt{|\widetilde{\mathbf{g}}|}\left\{\widehat{\mathbf{R}}[\widetilde{\mathbf{g}}]-\frac{1}{2}\widetilde{\mathbf{g}}_{\mu\nu}\mathbf{e}^{\nu}\varphi\ \mathbf{e}^{\mu}\varphi-\ {}^{\varphi}V(\varphi)\right\}, \qquad (28)
\end{aligned}
$$

with effective potential ${}^{\varphi}V(\varphi)$ (8) with for a gravitationally modified QC structure φ–field which for $(\Psi,\ \Upsilon)\leftrightarrow(\Phi,\widetilde{\Lambda})$ (17) defines N–adapted configurations of type (24) or (25). Such nonlinear transforms are possible only for generic off-diagonal cosmological solutions constructed using the AFDM. We shall write ${}^{\mathbf{q}}V({}^{\mathbf{q}}\varphi)$ for certain effective scalar like structures determined by a nontrivial QC configuration with $\mathbf{g}_{\mu\nu}\to\widetilde{\mathbf{g}}_{\mu\nu}$ and $\varphi = {}^{\mathbf{q}}\varphi$ described above.

In order to understand how actions of type (28) with effective free energy F (16), for DE, and ${}^{DM}F$ (27), for DM, encode conditions for inflation like in the Starobinsky quadratic gravity, let us consider small off–diagonal deformations of FLRW metrics to solutions of type (11) and (12). We introduce a new time like coordinate $\widehat{t}$, when $t=t(x^i,\widehat{t})$ and $\sqrt{|h_4|}\partial t/\partial\widehat{t}$, and a scale factor $\widehat{a}(x^i,\widehat{t})$ when the d–metric (1) can be represented in the form

$$
\begin{aligned}
ds^2 &= \widehat{a}^2(x^i,\widehat{t})[\eta_i(x^k,\widehat{t})(dx^i)^2+\widehat{h}_3(x^k,\widehat{t})(\mathbf{e}^3)^2-(\widehat{\mathbf{e}}^4)^2], \qquad (29)\\
\text{where } \eta_i &= \widehat{a}^{-2}e^{\psi},\widehat{a}^2\widehat{h}_3=h_3,\mathbf{e}^3=dy^3+\partial_k n\ dx^k,\widehat{\mathbf{e}}^4=d\widehat{t}+\sqrt{|h_4|}(\partial_i t+w_i).
\end{aligned}
$$

Using a small parameter ε, with $0\leq\varepsilon<1$, we model off–diagonal deformations if

$$
\eta_i\simeq 1+\varepsilon\chi_i(x^k,\widehat{t}),\partial_k n\simeq\varepsilon\widehat{n}_i(x^k),\sqrt{|h_4|}(\partial_i t+w_i)\simeq\varepsilon\widehat{w}_i(x^k,\widehat{t}). \qquad (30)
$$

This correspond to a subclass of generating functions, which for $\varepsilon\to 0$ result in $\Psi(t)$, or $\check{\Psi}(t)$, and, correspondingly $\Phi(t)$, and generating source $\Upsilon(t)$ in a form compatible to $\widehat{a}(x^i,\widehat{t})\to$

$\widehat{a}(t), \widehat{h}_3(x^i, \widehat{t}) \to \widehat{h}_3(\widehat{t})$ etc. Conditions of type (30) and homogeneous limits for generating functions and sources have to be imposed after a locally anisotropic solution (for instance, of type (12)), was constructed in explicit form. If we impose homogeneous conditions from the very beginning, we transform the (modified) Einstein equations with scalar filed in a nonlinear system of ODEs which do not describe gravitational and scalar field analogous quasicrystal structures. Applying the AFDM with generating and integration functions we solve directly nonlinear systems of PDEs and new classes of cosmological solutions are generated even in diagonal limits because of generic nonlinear and nonholonomic character of off–diagonal systems in MGFT. For $\varepsilon \to 0$ and $\widehat{a}(x^i, \widehat{t}) \to \widehat{a}(t)$, we obtain scaling factors which are very different from those in the FLRW cosmology with GR solutions. Nevertheless, we can mimic such cosmological models using redefined parameters and possible small off–diagonal deformations of cosmological evolution for MGTs as we explain in details in [22, 23, 24, 25]. In this work, we consider effective sources encoding contributions from the QC gravitational and scalar field structures, with

$$\widehat{a}^2 \widehat{h}_3 = \partial_t(\Phi^2)/\Upsilon[\Phi^2 - h_3^{[0]}(x^k)],$$

where $\partial_t(\Phi^2) = \widetilde{\Lambda}\ \partial_t(\Psi^2)/\Upsilon$, as follows respectively from formulas (18) and (17).

Nonhomogeneous QC structures with mixed discrete parameters and continuous degrees of freedom appear in a broader theoretical context related to quantum-gravity corrections and from the point of view of an exact renormalisation-group analysis. We omit such considerations in this work by note that inflation in our MGTs models can be generated for $1 \ll \varsigma$ and $\mathcal{M} \ll M_P$, which corresponds to an effective quasicrystal potential with magnitude ${}^{\mathbf{q}}V \ll M_P^4$, see details and similar calculations in [48]. In our approach, such values are for nontrivial QC configurations with diagonal limits. At certain nontrivial values ${}^{\mathbf{q}}\varphi$, when $\kappa^{-1}\ {}^{\mathbf{q}}\varphi$ are large compared to the Planck scale, a potential ${}^{\mathbf{q}}V = {}^{\varphi}V({}^{\mathbf{q}}\varphi)$ (8) is effectively sufficiently flat to produce phenomenologically acceptable inflation. In this model, the QC configuration determined by ${}^{\mathbf{q}}\varphi$ play the role of scalar field. This configuration determines a region with positive-definite Starobinsky potential where the term $\exp[-\sqrt{2/3}\ {}^{\mathbf{q}}\varphi]$ is dominant.

In general, a nontrivial QC gravitational and effective scalar configuration may result via generic off–diagonal parametric interactions described by solutions type (11) and (12) in effective potentials ${}^{\mathbf{q}}V$ with constants different from (8), with $Q \neq \varsigma^2, \varpi \neq 2$ and $P \neq \sqrt{2/3}$, when

$${}^{\mathbf{q}}V = Q(1 - \varpi e^{-P\ {}^{\mathbf{q}}\varphi} + ...),$$

where dots represent possible higher-order terms like $\mathcal{O}(e^{-2P\ {}^{\mathbf{q}}\varphi})$. This means that inflation can be generated by various types of effective quasicrystal structures which emphasizes the generality of the model. Possible cosmological implications of QCs can be computed following standard expressions in the slow-roll approximation for inflationary observables (we put left labels q in order to emphasize their effective QC origin). We have

$$\begin{aligned} {}^{\mathbf{q}}\epsilon &= \frac{M_P^2}{16\pi}\left(\frac{\partial\ {}^{\varphi}V/\partial\varphi}{{}^{\varphi}V}\mid_{{}^{\mathbf{q}}\varphi}\right)^2, \ {}^{\mathbf{q}}\eta = \frac{M_P^2}{8\pi}\frac{\partial^2\ {}^{\varphi}V/\partial^2\varphi}{{}^{\varphi}V}\mid_{{}^{\mathbf{q}}\varphi}, \\ {}^{\mathbf{q}}n_s &= 1 - 6\ {}^{\mathbf{q}}\epsilon + 2\ {}^{\mathbf{q}}\eta, \quad {}^{\mathbf{q}}r = 16\ {}^{\mathbf{q}}\epsilon. \end{aligned}$$

The a e-folding number for the inflationary phase

$$^{\mathbf{q}}N_{\star} = -\frac{8\pi}{M_P^2}\int\limits_{^{\mathbf{q}}\varphi_{(i)}}^{^{\mathbf{q}}\varphi_{(e)}} d\varphi \frac{^{\varphi}V}{\partial\ ^{\varphi}V/\partial\varphi}$$

with $^{\mathbf{q}}\varphi_{(i)}$ and $^{\mathbf{q}}\varphi_{(e)}$ being certain values of QC modifications at the beginning and, respectively, end of inflation. At leading order, considering the small quantity $e^{-P\ ^{\mathbf{q}}\varphi}$, one computes $^{\mathbf{q}}N_{\star} = e^{P\ ^{\mathbf{q}}\varphi}/P^2\varpi$ yielding

$$^{\mathbf{q}}n_s = 1 - 2P^2\varpi e^{-P\ ^{\mathbf{q}}\varphi} \simeq 1 - 2/\ ^{\mathbf{q}}N_{\star} \text{ and } \quad ^{\mathbf{q}}r = 8P^2\varpi^2 e^{-2P\ ^{\mathbf{q}}\varphi} \simeq 8/P^2\ ^{\mathbf{q}}N_{\star}^2.$$

In result, we get a proof that we can elaborate Starobinsky like scenarios using generic off-diagonal gravitational configurations (in GR and/or MGTs) determined by QC generating functions. For $^{\mathbf{q}}N_{\star} = 54 \pm 6$ for $P = \sqrt{2/3}$, we obtain characteristic predictions $^{\mathbf{q}}n_s \simeq 0.964$ and $^{\mathbf{q}}r \simeq 0.0041$ in a form highly consistent with the Plank data [1].

Finally, we note that for different QC configurations we may deviate from such characteristic MGTs predictions but still remain in GR via off-diagonal interactions resulting in QC structures. Such scenarios could not be involved in cosmology [6] even the authors [26, 27, 28] made substantial contributions both to the inflationary cosmology and physics of quasicrystals. The main problem was that nonlinearities and parametric off-diagonal interactions were eliminated from research from the very beginning in [3, 4, 5, 6] considering only of FLRW ansatz.

4.2 Reconstructing cosmological quasicrystal structures

We consider a model with Lagrange density (5) for $^{q}\mathbf{f}(\widehat{\mathbf{R}}) = \widehat{\mathbf{R}}^2 + \mathbf{M}(\ ^{q}\mathbf{T})$, where $^{q}\mathbf{T}$ is the trace of the energy–momentum tensor for an effective QC-structure determined by $(\mathbf{g}_{\alpha\beta}, \mathbf{D}_{\mu}, \varphi)$. Let us denote $^{q}\mathbf{M} := \partial\mathbf{M}/\partial\ ^{q}\mathbf{T}$ and $\widehat{H} := \partial_t\widehat{a}/\widehat{a}$ for a limit $\widehat{a}(x^i, \widehat{t}) \to \widehat{a}(t)$ in (29). In general, cosmological solutions are characterized by nonlinear symmetries (17) of generating functions and sources when the value $\widehat{a}(t)$ is different from $\mathring{a}(t)$ for a standard FLRW cosmology.

To taste the cosmological scenarios one considers the redshift $1 + z = \widehat{a}^{-1}(t)$ for a function $^{q}T = \ ^{q}T(z)$ and a new "shift" derivative when $\partial_t s = -(1+z)H\partial_z$,for instance, for a function $s(t)$. Following the method with nonholonomic variables elaborated in [24], we obtain for QC structures a set of three equations

$$\begin{aligned} 3\widehat{H}^2 + \frac{1}{2}[\ ^{q}\mathbf{f}(z) + \mathbf{M}(z)] - \kappa^2\rho(z) &= 0, \\ -3\widehat{H}^2 + (1+z)\widehat{H}(\partial_z\widehat{H}) - \frac{1}{2}\{\ ^{q}\mathbf{f}(z) + \mathbf{M}(z) + 3(1+z)\widehat{H}^2 &= 0, \\ \rho(z)\ \partial_z\ \mathbf{f} &= 0. \end{aligned} \tag{31}$$

Using transforms of type (17) for the generating function, we fix $\partial_z\ ^{q}\mathbf{M}(z) = 0$ and $\partial_z\ \mathbf{f} = 0$ which allows nonzero densities in certain adapted frames of references. The functional $\mathbf{M}(\ ^{q}\mathbf{T})$ encodes QC gravitational configurations for the evolution of the energy-density of type $\rho =$

$\rho_0 a^{-3(1+\vartheta)} = \rho_0(1+z)a^{3(1+\vartheta)}$ for the dust matter approximation with a constant ϑ and $\rho \sim (1+z)^3$.

Using (31), it is possible to elaborate reconstruction procedures for nontrivial QC configurations generalizing MGTs in nonholonomic variables. We can introduce the "e-folding" variable $\chi := \ln a/a_0 = -\ln(1+z)$ instead of the cosmological time t and compute $\partial_t s = \widehat{H}\partial_\chi s$ for any function s. In N-adapted frames, we derive the nonholonomic field equation corresponding to the first FLRW equation is

$$^q\mathbf{f}(\widehat{\mathbf{R}}) = (\widehat{H}^2 + \widehat{H}\ \partial_\chi \widehat{H})\partial_\chi[\ ^q\mathbf{f}(\widehat{\mathbf{R}})] - 36\widehat{H}^2\left[4\widehat{H} + (\partial_\chi\widehat{H})^2 + \widehat{H}\partial^2_{\chi\chi}\widehat{H}\right]\partial^2_{\chi\chi}\ ^q\mathbf{f}(\widehat{\mathbf{R}})] + \kappa^2\rho.$$

Introducing an effective quadratic Hubble rate, $\tilde{\kappa}(\chi) := \widehat{H}^2(\chi)$, where $\chi = \chi(\widehat{\mathbf{R}})$ for certain parameterizations, this equation transforms into

$$^q\mathbf{f} = -18\tilde{\kappa}(\widehat{\mathbf{R}})[\partial^2_{\chi\chi}\tilde{\kappa}(\chi) + 4\partial_\chi\tilde{\kappa}(\chi)]\frac{\partial^2\ ^q\mathbf{f}}{\partial\widehat{\mathbf{R}}^2} + 6\left[\tilde{\kappa}(\chi) + \frac{1}{2}\partial_\chi\tilde{\kappa}(\chi)\right]\frac{\partial\ ^q\mathbf{f}}{\partial\widehat{\mathbf{R}}} + 2\rho_0 a_0^{-3(1+\vartheta)}a^{-3(1+\vartheta)\chi(\widehat{\mathbf{R}})}. \tag{32}$$

Off-diagonal cosmological metrics encoding QC structures are of type (29) with $t \to \chi$, and a functional $^q\mathbf{f}(\widehat{\mathbf{R}})$ used for computing the generating source $\boldsymbol{\Upsilon}$ for prescribed generating function Φ. Such nonlinear systems can be described effectively by the field equations for an (nonholonomic) Einstein space $\widehat{\mathbf{R}}^\alpha{}_\beta = \widetilde{\Lambda}\delta^\alpha{}_\beta$. The functional $\partial\ ^q\mathbf{f}/\partial\widehat{\mathbf{R}}$ and higher functional derivatives vanish for any functional dependence $\mathbf{f}(\widetilde{\Lambda})$ because $\partial_\chi\widetilde{\Lambda} = 0$. The recovering procedure can be simplified substantially by using re-definitions of generating functions.

Let us consider an example with explicit reconstruction of MGT and nonholonomically deformed Einstein spaces with QC structure when the ΛCDM era can be reproduced. We chose any $\widehat{a}(\chi)$ and $\widehat{H}(\chi)$ for an off-diagonal (29). We obtain an analog of the FLRW equation for ΛCDM cosmology,

$$3\kappa^{-2}\widehat{H}^2 = 3\kappa^{-2}H_0^2 + \rho_0\widehat{a}^{-3} = 3\kappa^{-2}H_0^2 + \rho_0 a_0^{-3}e^{-3\chi},$$

where H_0 and ρ_0 are constant values. The effective quadratic Hubble rate and the modified scalar curvature, $\widehat{\mathbf{R}}$, are computed respectively,

$$\tilde{\kappa}(\zeta) := H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\chi} \text{ and } \widehat{\mathbf{R}} = 3\partial_\chi\tilde{\kappa}(\chi) + 12\tilde{\kappa}(\chi) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\chi}.$$

The equation (32) transforms into

$$X(1-X)\frac{\partial^2\ ^q\mathbf{f}}{\partial X^2} + [z_3 - (z_1+z_2+1)X]\frac{\partial\ ^q\mathbf{f}}{\partial X} - z_1 z_2\ ^q\mathbf{f} = 0,$$

for constants subjected to the conditions $z_1 + z_2 = z_1 z_2 = -1/6$ and $z_3 = -1/2$, when $3\chi = -\ln[\kappa^{-2}\rho_0^{-1}a_0^3(\widehat{\mathbf{R}} - 12H_0^2)]$ and $X := -3 + \widehat{\mathbf{R}}/3H_0^2$. The solutions of such nonholonomic QC equations with constant coefficients and for different types of scalar curvatures can be constructed similarly to [24]. In terms of Gauss hypergeometric functions, $^q\mathbf{f} = {}^qF(X) := {}^qF(z_1, z_2, z_3; X)$, we obtain

$$F(X) = KF(z_1, z_2, z_3; X) + BX^{1-z_3}F(z_1 - z_3 + 1, z_2 - z_3 + 1, 2 - z_3; X)$$

for some constants K and B. Such reconstructing formulas prove in explicit form that MGT and GR theories with QC structure encode ΛCDM scenarios without the need to postulate the existence of an effective cosmological constant. Such a constant can be stated by nonlinear transforms and redefinitions of the generating functions and (effective) energy momentum source for matter fields.

5 Quasicrystal Models for Dark Energy and Dark Matter

The modern cosmological paradigm is constructed following observational evidences that our Universe experiences an accelerating expansion [1]. Respectively, the dark energy, DE, and dark matter, DM, are considered to be responsible for acceleration and the dynamics of spiral galaxies. In order to solve this puzzle of gravity and particle physics and cosmology a number of approaches and MGTs were elaborated during last 20 years, see reviews and original results in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In a series of our recent works [22, 23, 24, 25], we proved that DE and DM effects can be modelled by generic off–diagonal gravitational and matter field interactions both in GR and MGTs. For models with QC structure, we do not need to "reconsider" the cosmological constant for gravitational field equations. We suppose that an effective $\widetilde{\Lambda}$ can be induced nonlinear symmetries of the generating functions and effective source which results in a QC Starobinsky like scenarios. In this section, we prove that QC structures can be also responsible for Universe acceleration and DE and DM effects.

5.1 Encoding off-diagonal QC structures into canonical d–torsions

It possible to reformulate the GR with the LC–connection in terms of an equivalent teleparallel theory with the Weizenboock connection (see, for instance, [49, 50, 51]) and study $f(T)$-theories of gravity, with T from torsion, which can be incorporated into a more general approach for various modifications of the gravitational Lagrangian $R \to f(R, T, F, L...)$. Such models can be integrated in very general forms for geometric variables of type $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. Off–diagonal configurations on GR and MGT with nonholonomic and aperiodic structures of QC or other type (noncommutative, fractional, diffusion etc.) ones can be encoded respectively into the torsion, $\widehat{\mathcal{T}}^{\alpha}$, and curvature, $\widehat{\mathcal{R}}^{\alpha}{}_{\beta}$, tensors. By definition, such values are defined and denoted respectively ${}^{q}\mathcal{T}^{\alpha} := \widehat{\mathcal{T}}^{\alpha}[\ {}^{q}\Psi]$ and ${}^{q}\mathcal{R}^{\alpha} := \widehat{\mathcal{R}}^{\alpha}[\ {}^{q}\Psi]$ in order to emphasize the QC structure of generating functions and sources. Such values can be computed in N-adapted form using the canonical d–connection 1–form ${}^{q}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}\mathbf{e}^{\gamma}$, where ${}^{q}\mathbf{D} = \{\ {}^{q}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}\}$,

$$
\begin{aligned}
{}^{q}\mathcal{T}^{\alpha} \ &: \ = \ {}^{q}\mathbf{D}\mathbf{e}^{\alpha} = d\mathbf{e}^{\alpha} + \ {}^{q}\mathbf{\Gamma}^{\alpha}{}_{\beta} \wedge \mathbf{e}^{\beta} = \ {}^{q}\mathbf{T}^{\alpha}{}_{\beta\gamma}\mathbf{e}^{\beta} \wedge \mathbf{e}^{\gamma} \text{ and} \\
{}^{q}\mathcal{R}^{\alpha}{}_{\beta} := \ & \ {}^{q}\mathbf{D}\ {}^{q}\mathbf{\Gamma}^{\alpha}{}_{\beta} = d\ {}^{q}\mathbf{\Gamma}^{\alpha}{}_{\beta} - \ {}^{q}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \ {}^{q}\mathbf{\Gamma}^{\alpha}{}_{\gamma} = \ {}^{q}\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta}.
\end{aligned}
$$

In such formulas, we shall omit the left label "q" and write, for instance, $\mathbf{D}$, $\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$, $\mathbf{T}^{\alpha}{}_{\beta\gamma}$ etc. if certain continuous limits are considered for the generating functions/sources and respective geometric objects. Hereafter we shall work with standard N–adapted canonical values of metrics, frames and connections which are generated by apperiodic QC $(\ {}^{q}\Psi,\ \ {}^{q}\Upsilon) \leftrightarrow (\ {}^{q}\Phi,\ {}^{q}\widetilde{\Lambda})$.

Such configurations are with a nontrivial effective ${}^{q}\mathcal{T}^{\alpha}$ induced nonholonomically. Even at the end we can extract LC–configurations by imposing additional nonholonomic constraints and integral sub-varieties with $({}^{q}\check{\Psi},\ {}^{q}\Upsilon) \leftrightarrow ({}^{q}\check{\Phi},\ {}^{q}\widetilde{\Lambda})$ all diagonal and off-diagonal cosmological solutions are determined by geometric and physical data encoded in $\{{}^{q}\mathbf{T}^{\alpha}{}_{\beta\gamma}\}$. For ${}^{q}\mathbf{D} \to {}^{q}\nabla$, the gravitational and matter field interactions are encoded into $\mathbf{e}^{\alpha}{}_{\alpha'}[\mathbf{g},\mathbf{N}]$ like in (1). In general, we can work in equivalent form with different type theories when

$$R \Longleftrightarrow \widehat{\mathbf{R}} \Longleftrightarrow f({}^{q}\mathbf{R}) \Longleftrightarrow f({}^{q}\mathcal{T})$$

are all completely defined by the same metric structure and data $(\mathbf{g},\mathbf{N})$. Here we note that ${}^{q}\mathbf{T}$ is constructed for the canonical d–connection ${}^{q}\mathbf{D}$ in a metric–affine spacetime with aperiodic order and this should be not confused with theories of type $f(R,T)$, where T is for the trace of the energy-momentum tensor.

We construct an equivalent $f({}^{q}\mathcal{T})$ theory for DE and DM configurations determined by a QC structure in this form: Let consider respectively the contorsion and quasi-contorsion tensors

$${}^{q}\mathbf{K}^{\mu\nu}{}_{\lambda} = \frac{1}{2}({}^{q}\mathbf{T}^{\mu\nu}{}_{\lambda} - {}^{q}\mathbf{T}^{\nu\mu}{}_{\lambda} + {}^{q}\mathbf{T}_{\lambda}{}^{\nu\mu}),\ {}^{q}\mathbf{S}_{\lambda}{}^{\nu\mu} = \frac{1}{2}({}^{q}\mathbf{K}^{\mu\nu}{}_{\lambda} + \delta^{\mu}_{\lambda}\,{}^{q}\mathbf{T}^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\lambda}\,{}^{q}\mathbf{T}^{\alpha\mu}{}_{\alpha})$$

for any ${}^{q}\mathcal{T}^{\alpha} = \{{}^{q}\mathbf{T}^{\mu\nu}{}_{\lambda}\}$. Then the canonical torsion scalar is defined ${}^{q}\mathcal{T} := {}^{q}\mathbf{T}^{\alpha}{}_{\beta\gamma}\,{}^{q}\mathbf{T}_{\alpha}{}^{\beta\gamma}$. The nonholonomic redefinition of actions and Lagrangians (5) and (28) in terms of ${}^{q}\mathcal{T}$ is

$$\mathcal{S} = \int d^4u \left[\frac{{}^{q}_{G}\mathcal{L}}{2\kappa^2} + {}^{m}\widehat{\mathcal{L}}\right] \tag{33}$$

where the Lagrange density for QC gravitational interactions is ${}^{q}_{G}\mathcal{L} = {}^{q}\mathcal{T} + f({}^{q}\mathcal{T})$.

The equations of motion in a flat FLRW universe derived for solutions o type (29) (for simplicity, we omit small parameter off-diagonal deformations (30)) are written in the form

$$\begin{aligned} 6H^2 + 12H^2 f^{\circ}({}^{q}\mathcal{T}) + f({}^{q}\mathcal{T}) &= 2\kappa^2\rho_{[i]}, \\ 2(2\partial_t H + 3H^2) + f({}^{q}\mathcal{T}) + 4(\partial_t H + 3H^2) f^{\circ}({}^{q}\mathcal{T}) - 48H^2(\partial_t H) f^{\circ\circ}({}^{q}\mathcal{T}) &= 2\kappa^2 p_{[i]}, \end{aligned}$$

where $f^{\circ} := df/d\ {}^{q}\mathcal{T}$ and $\rho_{[i]}$ and $p_{[i]}$ denote respectively the energy and pressure of a perfect fluid matter imbedded into a QC like gravitational and scalar matter type structure. Similar equations have been studied in Refs. [52, 53, 54, 55]. For $\kappa^2 = 8\pi G$, these equations can be written respectively as constraints equations

$$3H^2 = \rho_{[i]} + {}^{q}\rho, 2\partial_t H = -(\rho_{[i]} + p_{[i]} + {}^{q}\rho + {}^{q}p),$$

with additional effective QC type matter

$${}^{q}\rho = -6H^2 f^{\circ} - f/2,\ \ {}^{q}p = 2\partial_t H(1 + f^{\circ} - 12H^2 f^{\circ\circ}) + 6H^2 f^{\circ} + f/2. \tag{34}$$

Now we can elaborate our approach with DE and DM determined by aperiodic QC configurations of gravitational - scalar field systems.[3]

[3] Above equations can be written in a standard form for f–modified cosmology with

$${}^{ef}\Omega = \Omega_{[i]} + {}^{q}\Omega := \frac{\rho_{[i]}}{3H^2} + \frac{{}^{q}\rho}{3H^2} = 1,$$

with certain effective ${}^{ef}\rho = \rho_{[i]} + {}^{q}\rho$, ${}^{ef}p = p_{[i]} + {}^{q}p$ and ${}^{ef}\omega = {}^{ef}p/{}^{ef}\rho$ encoding an aperiodic QC order.

5.2 Interaction between DE and DM in aperiodic QC vacuum

In this section, we ignore all other forms of energy and matter and study how interact directly aperiodically QC structured DE and DM. Respective densities of QC dark energy and dark matter are parameterized

$$^{q}\rho = {}^{q}_{DE}\rho + {}^{q}_{DM}\rho \text{ and } {}^{q}p = {}^{q}_{DE}p + {}^{q}_{DM}p,$$

when (34) is written in the form

$$2\partial_t({}^{q}_{DE}\rho + {}^{q}_{DM}\rho) = (\partial_t \, {}^{q}\mathcal{T}) \left(f^{\circ} + 2 \, {}^{q}\mathcal{T} f^{\circ\circ}\right).$$

For perfect two fluid models elaborated in N–adapted form [22, 23, 24, 25], the interaction DE and DM equations are written

$$\begin{aligned} \kappa^2({}^{q}\rho + {}^{q}p) &= -2\partial_t H, \text{ subjected to} \\ \partial_t({}^{q}_{DE}\rho) + 3H({}^{q}_{DE}p + {}^{q}_{DE}\rho) &= -Q \text{ and } \partial_t({}^{q}_{DM}\rho) + 3H({}^{q}_{DM}p + {}^{q}_{DM}\rho) = Q. \end{aligned} \quad (35)$$

Above equations result in such a functional equation

$$2 \, {}^{q}\mathcal{T} f^{\circ\circ} + f^{\circ} + 1 = 0,$$

which can be integrated in trivial and nontrivial forms with certain integration constants C, C_0 and $C_1 = 0$ (this condition follows from (35)),

$$f({}^{q}\mathcal{T}) = \left\{ \begin{array}{c} -{}^{q}\mathcal{T} + C \\ -{}^{q}\mathcal{T} - 2C_0\sqrt{-{}^{q}\mathcal{T}} + C_1 \end{array} \right. .$$

So, the QC structure effectively contributes to DE and DM interaction via a nontrivial nonholonomically induced torsion structure. Such nontrivial aperiodic configurations exist via nontrivial C and C_0 even we impose the conditions ${}^{q}\mathcal{T}$ in order to extract certain diagonal LC–configurations. We note that in both cases of solutions for $f({}^{q}\mathcal{T})$ we preserve the conditions ${}^{ef}\Omega = 1$ and ${}^{ef}\omega = -1$.

5.3 Quasicrystal DE structures and matter sources

We analyse how aperiodic QC structure modify DE and DM and ordinary matter OM interactions and cosmological scenarios, see similar computations in [54, 55] but for a different type of torsion (for the Weitzenböck connection).

5.3.1 Interaction between DE and ordinary matter in gravitational QC media

Now, we model configuration when aperiodic DE interacts with OM (we use label "o" from ordinary, $({}^{o}\rho + {}^{o}p)$ for ${}^{q}\rho = {}^{q}_{DE}\rho$. We obtain such equations of interactions between DE and DM equations are written

$$\begin{aligned} \kappa^2({}^{q}_{DE}\rho + {}^{q}_{DE}p + {}^{o}\rho + {}^{o}p) &= -2\partial_t H, \text{ subjected to} \\ \partial_t({}^{q}_{DE}\rho) + 3H({}^{q}_{DE}p + {}^{q}_{DE}\rho) &= -Q \text{ and } \partial_t({}^{o}\rho) + 3H({}^{o}\rho + {}^{o}p) = Q. \end{aligned} \quad (36)$$

The equation (34) transform into

$$\partial_t({}^q_{DE}\rho) = (\partial_t \, {}^q\mathcal{T})({}^q\mathcal{T} f^{\circ\circ} + \frac{1}{2} f^{\circ}),$$

which together with above formulas result in

$$\partial_t({}^q_{DE}\rho + {}^o\rho + \frac{1}{2} \, {}^q\mathcal{T}) = 0.$$

For $f({}^q\mathcal{T})$, these formulas result in a second order functional equation

$$(2 \, {}^q\mathcal{T} f^{\circ\circ} + f^{\circ} + 1) = -2({}^o\rho)^{\circ}.$$

We can construct solutions of this equation by a splitting into two effective ODEs with a nonzero constant Z_0, when

$$f^{\circ\circ} + (2 \, {}^q\mathcal{T})^{-1} f^{\circ} = -Z_0 \text{ and } 2 \, {}^o\rho + 1 = 2Z_0 \, {}^q\mathcal{T}.$$

Such classes of solutions are determined by integration constants C_2 and $C_3 = 2C_4$ (this condition is necessary in order to solve (35)): for ${}^o\rho = -C_4 - {}^q\mathcal{T} + Z_0({}^q\mathcal{T})^2/2$, the aperiodic QC contribution is

$$f({}^q\mathcal{T}) = C_3 - 2C_2\sqrt{| - {}^q\mathcal{T}|} - Z_0({}^q\mathcal{T})^2/3.$$

We can chose $H_0 = 74.2 \pm 3.6 \frac{Km}{s} \frac{Mp}{c}$ and t_0 as the present respective Hubble parameter and cosmic time and state the current density of the dust $\rho(t_0) = {}^m\rho_0 = 3 \times 1.5 \times 10^{-67} eV^2$. For an arbitrary constant C_2, we get the gravitational action (35) and ${}^o\rho$ both modified by QC contributions via

$$C_4 = {}^m\rho_0 - 3H_0^2(1 - 6Z_0H_0^2).$$

The effective parameters of state

$${}^{ef}\omega = -({}^q\mathcal{T})^{-1}\{Z_0({}^q\mathcal{T})^2 + 4[1 - 2\partial_t H \; Z_0({}^q\mathcal{T})] + C_3\} \text{ and } {}^{ef}\Omega = 1$$

describe an universe dominated by QC dark energy interacting with ordinary matter.

5.3.2 Van der Waals fluid interacting with aperiodic DM

The state equation for such a fluid (with physical values labeled by w) is

$${}^wp(3 - {}^w\rho) + 8 \, {}^wp \, {}^w\rho - 3({}^w\rho)^2 = 0,$$

which results in the equations for interaction of the QC DE with such a van der Waals OM,

$$\begin{aligned} \kappa^2({}^q_{DE}\rho + {}^q_{DE}p + {}^w\rho + {}^wp) &= -2\partial_t H, \text{ subjected to} \\ \partial_t({}^q_{DE}\rho) + 3H({}^q_{DE}p + {}^q_{DE}\rho) &= -Q \text{ and } \partial_t({}^w\rho) + 3H({}^w\rho + {}^wp) = Q. \end{aligned}$$

Such equations are similar to (36) but with the OM pressure and density subjected to another state equation and modified DE interaction equations. The solutions the aperiodic QC contribution can be constructed following the same procedure with two ODEs and expressed for ${}^w\rho = -C_5 + Z_0({}^q\mathcal{T})^2/2, C_7 = 2C_5$, as

$$f({}^q\mathcal{T}) = C_7 + C_6\sqrt{| - {}^q\mathcal{T}|}/2 - {}^q\mathcal{T} - Z_0({}^q\mathcal{T})^2/3.$$

Taking $\partial_t H(t_0) = 0$ and ${}^{q}_{DE}p(t_0) + {}^{q}_{DE}\rho(t_0) = 0$, which constraints (see above equations) ${}^{w}p(t_0) + {}^{w}\rho(t_0) = 0$, and results in

$$C_5 = 3Z_0 H_0^2 + |74 - 96\ {}^{w}\omega|^{1/2} + \frac{5}{3},$$

for typical values ${}^{w}\omega = 0.5$ and $E = 10^{-10}$.

5.3.3 Chaplygin gas and DE - QC configurations

Another important example of OM studied in modern cosmology (see, for instance, [56]) is that of Chaplygin, ch, gas characterized by an equation of state ${}^{ch}p = -Z_1/\ {}^{ch}\rho$, for a constant $Z_1 > 0$. The corresponding equations for interactions between DE and such an OM is given by

$$\begin{aligned} \kappa^2({}^{q}_{DE}\rho + {}^{q}_{DE}p + {}^{ch}\rho + {}^{ch}p) &= -2\partial_t H, \text{ subjected to} \\ \partial_t({}^{q}_{DE}\rho) + 3H({}^{q}_{DE}p + {}^{q}_{DE}\rho) &= -Q \text{ and } \partial_t({}^{ch}\rho) + 3H({}^{ch}\rho + {}^{ch}p) = Q. \end{aligned}$$

The solutions for this system can be written for ${}^{ch}\rho = C_8 + Z_0({}^{q}\mathcal{T})^2/2, C_{10} = 2C_8$, as

$$f({}^{q}\mathcal{T}) = C_{10} + C_9\sqrt{|-\ {}^{q}\mathcal{T}|}/2 - {}^{q}\mathcal{T} - Z_0({}^{q}\mathcal{T})^2/3.$$

Let us assume $\partial_t H(t_0) = 0$ and ${}^{q}_{DE}p(t_0) + {}^{q}_{DE}\rho(t_0) = 0$, which results in ${}^{ch}p(t_0) + {}^{ch}\rho(t_0) = 0$ and

$$C_8 = 18Z_0 H_0^4 + |Z_1 - 9Z_0 H_0^4(1 + 36Z_0 H_0^4)|^{1/2},$$

for typical values $Z_1 = 1$ and $E = 10^{-10}$.

The solutions for different type of interactions of QC like DE and DM with OM subjected to corresponding equations of state (for instance, of van der Waals or Chaplygin gase) prove that aperiodic spacetime strucutres result, in general, in off–diagonal cosmological scenarious which in diagonal limits result in effects related directly to terms containing contributions of nonholonomically induces torsion. If the constructions are redefined in coordinate type variables, such terms transform into certain generic off–diagonal coefficients of metrics.

6 Discussion and Conclusions

This paper is devoted to the study of aperiodic quasicrystal, QC, like gravitational and scalar field structures in acceleration cosmology. It apply certain geometric methods for constructing exact solutions in mathematical cosmology. The main conclusion is that exact solutions with aperiodic order in modified gravity theories, MGTs, and general relativity, GR, and with generic off–diagonal metrics, confirm but also offer interesting alternatives to the original Starobinsky model. Here we emphasize that our work concerns possible spacetime aperiodic order and QC like discrete and continuous configurations at cosmological scales. This is different from the vast majority of QCs (discovered in 1982 [57], which attracted the Nobel prize for chemistry in 2011) are made from metal alloys. There are also examples of QCs found in nanoparticles and soft-matter systems with various examples of block copolymers etc. [58, 59, 60, 61, 62]. In a

complimentary way, it is of special interest to study configurations with aperiodic order present also interest in astronomy and cosmology, when a number of observational data confirm various type filament and deformed QC structures, see [63].

In this work, the emergence of aperiodic ordered structure in acceleration cosmology is investigated following geometric methods of constructing exact and parametric solutions in modified gravity theories, MGTs, and in general relativity, GR (such methods with applications in modern cosmology are presented in [19, 22, 23, 24]). For instance, QC configurations may be determined by generating functions encoding, for instance, a "golden rotation" of $\arccos(\tau^2/2\sqrt{2}) \approx 22.2388^\circ$ (where the golden ration is given by $\tau = \frac{1}{2}(1+\sqrt{5})$), see [36, 37]. The reason to use such aperiodic and discrete parameterized generating functions and effective sources of matter is that various cosmological scales can be reproduced as certain nonholonomic deformation and diffusion processes from a chosen QC configuration. The priority of our geometric methods is that we can work both with continuous and discrete type generating functions which allows to study various non-trivial deformed networks with various bounds and lengths re-arranging and deforming, for instance, icosahedral arrangements of tetrahedra etc.

The aperiodic QC gravity framework proves to be very useful, since many geometric and cosmological evolution scenarios can be realized in the context of this approach.The question is can such models be considered as viable ones in order to explain alternatively Starobinsky-lyke scenarious and provide a physical ground for explicit models of dark energy, DE, and dark matter, DM. Working with arbitrary generating functions and effective source it seems that with such MGTs everything can be realized and certain lack of predictibility is characteristic. From our point of view, the anholonomic frame deformation method, AFDM, is more than a simple geometric methods for constructing exact solutions for certain classes of important nonlinear systems of partial differential equations, PDEs, in mathematical relativity. It reflects new and former un-known properties and nonlinear symmetries of (modified) Einstein equations when generic off–diagonal interactions and mixed continuous and discrete structures are considered for vacuum and non-vacuum gravitational configurations. The AFDM is appealing in some sense to be "economical and very efficient" because allows to treat in the same manner by including fractional / random / noncommutative sources and respective interaction parameters. We can speculate on existence of noncommutative and/or nonassociative QC generalized structures in the framework of classical MGTs. Moreover, certain compatibility with the cosmological observational data can be achieved and, in addition, there are elaborated realistic models of geometric flows and grow of QC related to accelerating cosmology.

An interesting and novel research steam is related to the possibility to encode aperiodic QC structures into certain nonholonomic and generic off–diagonal metric configurations with nonholonomically induced canonical torsion fields. Such alternatives to the teleparallel and other MGTs equivalents of the GR allows to elaborate in a most "economic" way on QC models for DE and DM and study "aperiodic dark" interactions with ordinary matter (like van der Vaals and Chapliygin gas). A procedure which allows to reconstruct QCs is outlined following our former nonholonomic generalizations [22, 23, 24]. However, QC structures are characterised by Lypunov type functionals for free energy which for geometric models of gravity are related to certain generalized Perelman's functionals studied in [39]. Following such an approach, a new theory with aperiodic geometric originating DE and DM with generalized Ricci flows with

(non) holonomic/ commutative / fractional structures has to be elaborated and we defer such issues to our future work [38].

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[63] See, for instance, webpage: www.crystalinks.com/darkmatter.html