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REVIEW

A Clifford algebra-based grand unification program of gravity and the Standard Model: a review study

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Abstract: A Clifford $Cl(5, C)$ unified gauge field theory formulation of conformal gravity and $U(4) \times U(4) \times U(4)$ Yang–Mills in 4D, is reviewed along with its implications for the Pati–Salam (PS) group $SU(4) \times SU(2)_L \times SU(2)_R$, and *trinification* grand unified theory models of three fermion generations based on the group $SU(3)_C \times SU(3)_L \times SU(3)_R$. We proceed with a brief review of a unification program of 4D gravity and $SU(3) \times SU(2) \times U(1)$ Yang–Mills emerging from 8D pure quaternionic gravity. A realization of E_8 in terms of the $Cl(16) = Cl(8) \otimes Cl(8)$ generators follows, as a preamble to F. Smith's E_8 and $Cl(16) = Cl(8) \otimes Cl(8)$ unification model in 8D. The study of chiral fermions and instanton backgrounds in $\mathbf{CP}^2$ and $\mathbf{CP}^3$ related to the problem of obtaining three fermion generations is thoroughly studied. We continue with the evaluation of the coupling constants and particle masses based on the geometry of bounded complex homogeneous domains and geometric probability theory. An analysis of neutrino masses, Cabbibo–Kobayashi–Maskawa quark-mixing matrix parameters, and neutrino-mixing matrix parameters follows. We finalize with some concluding remarks about other proposals for the unification of gravity and the Standard Model, like string, M, and F theories and noncommutative and nonassociative geometry.

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Résumé : Nous passons en revue une formulation en théorie du champ unifié de Clifford $Cl(5, C)$ de la gravité conforme et de Yang–Mills $U(4) \times U(4) \times U(4)$ en 4D, avec ses implications pour le groupe de Pati-Salam $SU(4) \times SU(2)_L \times SU(2)_R$ et les modèles GUT de *trinification* de trois générations de fermions basés sur le groupe $SU(3)_C \times SU(3)_L \times SU(3)_R$. Nous commençons avec une brève revue du programme d'unification de la gravité 4D et Yang–Mills $SU(3) \times SU(2) \times U(1)$ qui émerge de la gravité quaternionique pure en 8D. Nous poursuivons avec une réalisation de E_8 en termes des générateurs de $Cl(16) = Cl(8) \otimes Cl(8)$, comme introduction au modèle d'unification de F. Smith E_8 et $Cl(16) = Cl(8) \otimes Cl(8)$ en 8D. Nous étudions en profondeur les fermions chiraux et les fonds d'instantons dans $\mathbf{CP}^2$ et $\mathbf{CP}^3$ reliés à la difficulté d'obtenir trois générations de fermions. Nous continuons avec l'évaluation des constantes de couplage et des masses des particules, sur la base de la géométrie des domaines homogènes complexes bornés et de la théorie de probabilité géométrique. Suit une étude des masses des neutrinos, des paramètres de la matrice de mélange de Cabbibo–Kobayashi–Maskawa et des paramètres de la matrice d'oscillation de neutrinos. Nous concluons avec des remarques sur les propositions faites pour unifier la gravité et le modèle standard, comme la théorie des cordes, la théorie M et F, et la géométrie non associative et non commutative. [Traduit par la Rédaction]

1. Introduction

Clifford, division, exceptional, and Jordan algebras are deeply related and essential tools in many aspects in physics [1–3]. Grand unification (GUT) models in 4D based on the exceptional E_8 Lie algebra have been studied for some time [4, 5]. The supersymmetric E_8 model has more recently been studied as a fermion family and GUT model [6]. The low-energy phenomenology of superstring-inspired E_6 models has been reviewed by Hewett and Rizzo [7]. Lisi [8] proposed a E_8 unification model with gravity, but it was plagued by many problems and criticisms. Another controversial and problematic model was the $E_8 \times E_8$ model of Traintaphyllou [9].

Supersymmetric nonlinear σ models of Kahler coset spaces $E_8/[SO(10) \times SU(3) \times U(1)]$; $E_7/SU(5)$; $E_6/[SO(10) \times U(1)]$ are known to contain three generations of quarks and leptons as (quasi) Nambu–Goldstone *superfields* (see ref. 10 and references therein). The coset model based on $G = E_8$ gives rise to three left-handed generations assigned to the **16** multiplet of $SO(10)$, and one right-handed generation assigned to the $\mathbf{16}^*$ multiplet of $SO(10)$. The coset model based on $G = E_7$ gives rise to three generations of quarks and leptons assigned to the $\mathbf{5}^* + \mathbf{10}$ multiplets of $SU(5)$, and a Higgsino (the fermionic partner of the scalar Higgs) in the **5** representation of $SU(5)$.

A Chern–Simons E_8 gauge theory of gravity, based on the octic E_8 invariant construction by Cederwall and Palmkvist [11], was proposed [12] as a unified field theory (at the Planck scale) of a Lanczos–Lovelock gravitational theory with a E_8 generalized Yang–Mills field theory, which is defined in the 15D boundary of a 16D bulk space. The role of the Clifford algebra $Cl(16)$ associated with a 16D bulk was essential [12]. In particular, it was discussed how an E_8 Yang–Mills in 8D, after a sequence of symmetry breaking processes based on the *noncompact* forms of exceptional groups as follows $E_{8(-24)} \to E_{7(-5)} \times SU(2) \to E_{6(-14)} \times SU(3) \to SO(8, 2) \times U(1)$, leads to a conformal gravitational theory in 8D based on gauging the noncompact conformal group $SO(8, 2)$ in 8D. Performing a Kaluza–Klein–Batakis [13] compactification on CP^2, involving a nontrivial *torsion* that bypasses the no-go theorems that one cannot obtain $SU(3) \times SU(2) \times U(1)$ from a Kaluza–Klein mechanism in 8D, leads to a conformal gravity – Yang–Mills unified theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in 4D.

An interesting comparison between the number of physical (helicity) states of the minimal supersymmetric Standard Model,

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the Clifford algebra $Cl(8)$, and the unique and exceptional self-dual 24-cell polytope in four dimensions, the octa-cube, was analyzed by Boya [14], who found that the total number of bosonic and fermionic degrees of freedom was 256, which is the dimension of the $Cl(8)$ algebra. Another interesting numerical coincidence is that if one assumes that the neutrino is massive, each massive fermion generation in 4D is comprised of 16 fermions. A Dirac spinor in 4D has four complex components, that is, eight real components, hence the total number of real components is then $16 \times 8 = 128$. The *mirror* fermions yield another 128 real components, so the total number of degrees of freedom for one generation plus one antigeneration (mirror fermions) is 256, which coincides also with the dimension of the $Cl(8)$ algebra. We also may notice that a $Cl(16)$ spinor with $2^{16/2} = 256$ components in 16D can be decomposed into spinors of positive and negative chirality with 128 components, respectively.

A candidate action for an exceptional E_8 gauge theory of gravity in 8D was constructed [15]. It was obtained by recasting the E_8 group as the semidirect product of $GL(8, R)$ with a deformed Weyl–Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of ranks two and three. Other actions were proposed, like the quartic E_8 group-invariant action in 8D associated with the Chern–Simons E_8 gauge theory defined on the seven-dimensional boundary of an 8D bulk. The E_8 gauge theory of gravity can be embedded into a more general extended gravitational theory in Clifford spaces (C-spaces) associated with the Clifford $Cl(16)$ algebra because $E_8 \subset Cl(8) \otimes Cl(8) = Cl(16)$.

The aim of this work is to review a Clifford algebra - based GUT program of gravity and the Standard Model. Section 2 is devoted to a thorough study of Clifford algebras, conformal gravity and $U(4) \times U(4) \times U(4)$ Yang–Mills unification [16]. It includes: (*i*) a Clifford algebra realization of the conformal group $SO(4, 2)$, $U(4)$ and how the pseudo-unitary algebras $U(p, q)$ can be obtained from the unitary ones $U(p + q)$ via the Weyl unitary trick; (*ii*) a study of gravity, trinification, and PS models from $Cl(5, C)$ gauge field theories; (*iii*) an embedding of $U(4)$ into $SO(8) \subset Cl(8)$ via the use of fermionic oscillator algebras will allow us to end the group-chain with $SO(10)$, which is a GUT group candidate, because it admits complex representations to describe chiral fermions in 4D.

In Sect. 3 we briefly review how 4D Gravity and $SU(3) \times SU(2) \times U(1)$ Yang–Mills emerges from 8D quaternionic gravity [17]. A realization of E_8 in terms of $Cl(16) = Cl(8) \otimes Cl(8)$ generators follows in Sect. 4. In Sect. 5 a detailed analysis of the incorporation of fermions is presented.

Section 6 is devoted to Smith's $E_8 \subset Cl(8) \otimes Cl(8)$ algebra-based unification model in 8D. The Coleman–Mandula theorem and gauge bosons as fermion condensates are discussed along with an octonionic realization of $GL(8, R)$ [18] and the $SU(3)$ colour algebra of quarks [19]. We proceed with the Lagrangian construction in Smith's physics model and an extensive analysis of chiral fermions, number of generations, and instanton backgrounds in CP^n based on the work by Dolan and Nash [20].

In Sect. 7 a detailed study of complex geometric domains, couplings, masses, and parameters of the Standard Model is presented. The evaluation of the fine structure constant by Wyler [21], and the weak and strong couplings by Smith [22, 23] are performed via the geometric probability theory formalism analysis as described explicitly by Castro [24]. The particle masses, electroweak bosons, Higgs mass, the lepton and quark masses, the Cabibbo–Kobayashi–Maskawa parameters, the neutrino masses and neutrino-mixing (Pontecorvo–Maki–Nakagawa–Sakata) matrix parameters are obtained following the construction of Smith [22, 23]. We also include a discussion of the lepton masses procedure by Gonzalez-Martin [25]. Section 7 ends with a discussion on the other approaches to obtain physical constants, like the one by Beck [26].

To conclude in Sect. 8, we add some important remarks related to string (M, F) theory and noncommutative and nonassociative geometry.

2. Clifford algebras and conformal gravity, $U(4) \times U(4) \times U(4)$ Yang–Mills unification

2.1. A Clifford algebra realization of the conformal group $SO(4, 2)$

The aim of this section is to explain the relationship between Clifford-algebra-valued gauge field theories and conformal gravity [16]. By fixing some of the gauge symmetries and imposing some constraints one recovers ordinary gravity. We shall begin by showing how the conformal algebra in four dimensions admits a Clifford algebra realization; that is, the generators of the conformal algebra can be expressed in terms of the Clifford algebra basis generators. The conformal algebra in four dimensions $so(4, 2)$ is isomorphic to $su(2, 2)$.

Let $\eta_{ab} = (-, +, +, +)$ be the Minkowski space–time (flat) metric in $D = 3 + 1$ dimenisons. The epsilon tensors are defined as $\varepsilon_{0123} = -\varepsilon^{0123} = 1$, The real Clifford $Cl(3, 1, R)$ algebra associated with the tangent space of a 4D space–time $\mathcal{M}$ is defined by the anticommutators

$$\{\Gamma_a, \Gamma_b\} \equiv \Gamma_a\Gamma_b + \Gamma_b\Gamma_a = \; = 2\eta_{ab} \tag{2.1a}$$

such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab} \qquad \Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 \\ (\Gamma_5)^2 = 1 \qquad \{\Gamma_5, \Gamma_a\} = 0 \tag{2.1b}$$

$$\Gamma_{abcd} = \epsilon_{abcd}\Gamma_5 \qquad \Gamma_{ab} = \frac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a) \tag{2.2a}$$

$$\Gamma_{abc} = \epsilon_{abcd}\Gamma_5\Gamma^d \qquad \Gamma_{abcd} = \epsilon_{abcd}\Gamma_5 \tag{2.2b}$$

$$\Gamma_a\Gamma_b = \Gamma_{ab} + \eta_{ab} \qquad \Gamma_{ab}\Gamma_5 = \frac{1}{2}\epsilon_{abcd}\Gamma^{cd} \tag{2.2c}$$

$$\Gamma_{ab}\Gamma_c = \eta_{bc}\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\Gamma_5\Gamma^d \tag{2.2d}$$

$$\Gamma_c\Gamma_{ab} = \eta_{ac}\Gamma_b - \eta_{bc}\Gamma_a + \epsilon_{abcd}\Gamma_5\Gamma^d \tag{2.2e}$$

$$\Gamma_a\Gamma_b\Gamma_c = \eta_{ab}\Gamma_c + \eta_{bc}\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\Gamma_5\Gamma^d \tag{2.2f}$$

$$\Gamma^{ab}\Gamma_{cd} = \epsilon^{ab}_{cd}\Gamma_5 - 4\delta^{[a}_{[c}\Gamma^{b]}_{d]} - 2\delta^{ab}_{cd} \tag{2.2g}$$

$$\delta^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c\right) \tag{2.2h}$$

the generators Γ_{ab}, Γ_{abc}, and Γ_{abcd} are defined as usual by a signed-permutation sum of the antisymmetrizated products of the gammas. A representation of the $Cl(3, 1)$ algebra exists where the generators

$$\mathbf{1};\ \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 = -i\Gamma_0 \quad \text{and} \quad \Gamma_5 \tag{2.3}$$

are Hermitian; while the generators $\Gamma_a\Gamma_5$ and Γ_{ab} for $a, b = 1, 2, 3, 4$ are anti-Hermitian. Equations (2.1)–(2.3) make it possible to write the $Cl(3, 1)$ algebra-valued one-form as

$$\mathbf{A} = \left(a_\mu\mathbf{1} + b_\mu\Gamma_5 + e^a_\mu\Gamma_a + f^a_\mu\Gamma_a\Gamma_5 + \frac{1}{4}\omega^{ab}_\mu\Gamma_{ab}\right)dx^\mu \tag{2.4}$$

The physical significance of the field components a_μ, b_μ, e^a_μ, f^a_μ, ω^{ab}_μ in (2.4) will be explained later.

The Clifford-valued gauge field A_μ transforms according to $A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1}FU$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$F_{\mu\nu} = F^1_{\mu\nu}\mathbf{1} + F^5_{\mu\nu}\Gamma_5 + F^a_{\mu\nu}\Gamma_a + F^{a5}_{\mu\nu}\Gamma_a\Gamma_5 + \frac{1}{4}F^{ab}_{\mu\nu}\Gamma_{ab} \tag{2.5}$$

the Clifford-algebra-valued two-form field strength is $F = (1/2)F_{\mu\nu}dx^\mu \wedge dx^\nu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ where $\partial_\mu A_\nu = (\partial A_\nu/\partial x^\mu)$. The field-strength components are given by

$$F^1_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{2.6a}$$

$$F^5_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu + 2e^a_\mu f_{\nu a} - 2e^a_\nu f_{\mu a} \tag{2.6b}$$

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2f^a_\mu b_\nu - 2f^a_\nu b_\mu \tag{2.6c}$$

$$F^{a5}_{\mu\nu} = \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2e^a_\mu b_\nu - 2e^a_\nu b_\mu \tag{2.6d}$$

$$F^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega^b_{\nu c} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \leftrightarrow \nu \tag{2.6e}$$

At this stage we may provide the relation among the $Cl(3, 1)$ algebra generators and the conformal algebra $so(4, 2) \sim su(2, 2)$ in 4D. The operators of the conformal algebra can be written in terms of the Clifford algebra generators as [27]

$$\begin{aligned} &P_a = \frac{1}{2}\Gamma_a(1 - \Gamma_5) \qquad K_a = \frac{1}{2}\Gamma_a(1 + \Gamma_5) \qquad D = -\frac{1}{2}\Gamma_5 \\ &L_{ab} = \frac{1}{2}\Gamma_{ab} \end{aligned} \tag{2.7}$$

where P_a (a = 1, 2, 3, 4) are the translation generators; K_a are the conformal boosts; D is the dilation generator; and L_{ab} are the Lorentz generators. The total number of generators is respectively 4 + 4 + 1 + 6 = 15. From this realization of the conformal algebra generators (2.7), the explicit evaluation of the commutators yields

$$\begin{aligned} &[P_a, D] = P_a \qquad [K_a, D] = -K_a \qquad [P_a, K_b] = -2g_{ab}D + 2L_{ab} \\ &[P_a, P_b] = 0 \qquad [K_a, K_b] = 0\ldots \end{aligned} \tag{2.8}$$

which is consistent with the $su(2, 2) \sim so(4, 2)$ commutation relations. We should notice that the K_a, P_a generators in (2.7) are both comprised of Hermitian Γ_a *and* anti-Hermitian $\pm\Gamma_a\Gamma_5$ generators, respectively. The dilation D operator is Hermitian, while the Lorentz generator L_{ab} is anti-Hermitian. The fact that Hermitian and anti-Hermitian generators are required is consistent with the fact that $U(2, 2)$ is a pseudo-unitary group as we shall see later.

Having established this, one can infer that the real-valued tetrad V^a_μ field (associated with translations) and its real-valued partner $\tilde{V}^a_\mu$ (associated with conformal boosts) can be defined in terms of the real-valued gauge fields e^a_μ, f^a_μ as follows:

$$e^a_\mu\Gamma_a + f^a_\mu\Gamma_a\Gamma_5 = V^a_\mu P_a + \tilde{V}^a_\mu K_a \tag{2.9}$$

From (2.7) one learns that (2.9) leads to

$$\begin{aligned} &e^a_\mu - f^a_\mu = V^a_\mu \qquad e^a_\mu + f^a_\mu = \tilde{V}^a_\mu \\ &\Rightarrow e^a_\mu = \frac{1}{2}\left(V^a_\mu + \tilde{V}^a_\mu\right) \qquad f^a_\mu = \frac{1}{2}\left(\tilde{V}^a_\mu - V^a_\mu\right) \end{aligned} \tag{2.10}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given, respectively, by the linear combinations of (2.6c) and (2.6d)

$$\begin{aligned} &F^a_{\mu\nu} - F^{a5}_{\mu\nu} = \tilde{F}^a_{\mu\nu}[P] \qquad F^a_{\mu\nu} + F^{a5}_{\mu\nu} = \tilde{F}^a_{\mu\nu}[K] \\ &\Rightarrow F^a_{\mu\nu}\Gamma_a + F^{a5}_{\mu\nu}\Gamma_a\Gamma_5 = \tilde{F}^a_{\mu\nu}[P]P_a + \tilde{F}^a_{\mu\nu}[K]K_a \end{aligned} \tag{2.11a}$$

Inserting the expressions for e^a_μ, f^a_μ in terms of the vielbein V^a_μ and $\tilde{V}^a_\mu$ given by (2.10), yields the standard expressions for the torsion and conformal-boost curvature, respectively

$$\begin{aligned} &\tilde{F}^a_{\mu\nu}[P] = \partial_{[\mu}V^a_{\nu]} + \omega^{ab}_{[\mu}V_{\nu]b} - V^a_{[\mu}b_{\nu]} \\ &\tilde{F}^a_{\mu\nu}[K] = \partial_{[\mu}\tilde{V}^a_{\nu]} + \omega^{ab}_{[\mu}\tilde{V}_{\nu]b} + 2\tilde{V}^a_{[\mu}b_{\nu]} \end{aligned} \tag{2.11b}$$

The Lorentz curvature in (2.6e) can be recast in the standard form as

$$F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} = \partial_{[\mu}\omega^{ab}_{\nu]} + \omega^{ac}_{[\mu}\omega^b_{\nu]c} + 2\left(V^a_{[\mu}\tilde{V}^b_{\nu]} + \tilde{V}^a_{[\mu}V^b_{\nu]}\right) \tag{2.11c}$$

The components of the curvature corresponding to the Weyl dilation generator given by $F^5_{\mu\nu}$ in (2.6b) can be rewritten as

$$F^5_{\mu\nu} = \partial_{[\mu}b_{\nu]} + \frac{1}{2}\left(V^a_{[\mu}\tilde{V}_{\nu]a} - \tilde{V}^a_{[\mu}V_{\nu]a}\right) \tag{2.11d}$$

and the Maxwell curvature is given by $F^1_{\mu\nu}$ in (2.6a). A rescaling of the vielbein V^a_μ/l and $\tilde{V}^a_\mu/l$ by a length scale parameter, l, is necessary to endow the curvatures and torsion in (2.11a)–(2.11d) with the proper dimensions of *length*$^{-2}$ and *length*$^{-1}$, respectively.

To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\tilde{V}^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \mp f^a_\mu$ associated with the Γ_a, $\Gamma_a\Gamma_5$ generators of the Clifford algebra $Cl(3, 1)$ of the tangent space of space–time $\mathcal{M}^4$ after performing a Wick rotation $-i\Gamma_0 = \Gamma_4$.

Gauge invariant actions involving Yang–Mills terms of the form $\int\mathrm{Tr}(F \wedge {}^*F)$ and theta terms of the form $\int\mathrm{Tr}(F \wedge F)$ are straightforwardly constructed. For example, a $SO(4, 2)$ gauge-invariant action for conformal gravity is [28]

$$S = \int d^4x \epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma} R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma} \tag{2.12}$$

where the components of the Lorentz curvature two-form $R^{ab}_{\mu\nu}dx^\mu \wedge dx^\nu$ are given by (2.11c) after rescaling the vielbein V^a_μ/l and $\tilde{V}^a_\mu/l$ by a length scale parameter, l, to endow the curvature with the proper dimensions of *length*$^{-2}$.

The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2\xi^a e_{a\mu} = -2\xi_\mu$; that is, the parameter ξ_μ has the same number of degrees of freedom as b_μ. After further fixing the dilational gauge symmetry, setting the torsion to zero (which constrains the spin connection $\omega^{ab}_\mu(V^a_\mu)$ to be of the Levi–Civita form given by a function of the vielbein V^a_μ), and eliminating the $\tilde{V}^a_\mu$ field algebraically via its (nonpropagating) equations of motion [29], the expression in (2.12) leads to the de Sitter group $SO(4, 1)$ invariant Macdowell–Mansouri–Chamseddine–West (MMCW) action [30, 31] (suppressing space–time indices for convenience)

$$S = \int d^4x\left(R^{ab}(\omega) + \frac{1}{l^2}V^a \wedge V^b\right) \wedge \left(R^{cd}(\omega) + \frac{1}{l^2}V^c \wedge V^d\right)\epsilon_{abcd} \tag{2.13}$$

The action (2.13) is comprised of three terms. One term is the topological invariant Gauss–Bonnet term $R^{ab}(\omega) \wedge R^{cd}(\omega)\epsilon_{abcd}$. The standard Einstein–Hilbert gravitational action term is given by

$-(1/l^2)R^{ab}(\omega) \wedge V^c \wedge V^d\epsilon_{abcd}$, and the cosmological constant term $(1/l^4)V^a \wedge V^b \wedge V^c \wedge V^d\epsilon_{abcd}$. Parameter l is the de Sitter space's throat size; that is, l^2 is proportional to the square of the Planck scale (the Newtonian coupling constant).

The familiar Einstein–Hilbert gravitational action can also be obtained from a coupling of gravity to a scalar field like it occurs in a Brans–Dicke–Jordan theory of gravity

$$S = \frac{1}{2}\int d^4x\sqrt{g}\phi\left(\frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}g^{\mu\nu}D^c_\mu\phi\right) + b^\mu\left(D^c_\mu\phi\right) + \frac{1}{6}R\phi\right) \tag{2.14a}$$

where the conformally covariant derivative acting on a scalar field, ϕ, of Weyl weight one is

$$D^c_\mu\phi = \partial_\mu - b_\mu\phi \tag{2.14b}$$

Fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting ϕ = constant leads to the Einstein–Hilbert action for ordinary gravity.

To finalize this section, recall that gravity involves invariance under *diffeomorphisms* (coordinate transformations) and that gravitons ($g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab}$) have spin 2, not 1. What occurs is that the torsion constraint $R^a_{\mu\nu} = 0$ allows conversion of a combination of translations and Lorentz and dilation transformations of the vielbein e^a_μ into general coordinate transformations of the vielbein [32]. Nevertheless we must emphasize that gravity is *not* an ordinary gauge theory. If it were, we would have been able to quantize it long ago.

2.2. A Clifford algebra realization of $U(4)$

To obtain the generators of the compact $U(4) = SU(4) \times U(1)$ unitary group, in terms of the $Cl(3, 1)$ generators, a *different* basis involving a full set of Hermitian generators must be chosen of the form [16]

$$M_a = \frac{1}{2}\Gamma_a(1 - i\Gamma_5) \qquad N_a = \frac{1}{2}\Gamma_a(1 + i\Gamma_5) \qquad \mathcal{D} = \frac{1}{2}\Gamma_5$$
$$\mathcal{L}_{ab} = -\frac{i}{2}\Gamma_{ab} \tag{2.15}$$

One may choose, instead, a full set of anti-Hermitian generators by multiplying every generator M_a, N_a, $\mathcal{D}$, $\mathcal{L}_{ab}$ by $\mathbf{i}$ in (2.15), if one wishes. The choice (2.15) leads to a *different* algebra $so(6) \sim su(4)$ and whose commutators *differ* from those in (2.8)

$$[M_a, \mathcal{D}] = iN_a \qquad [N_a, \mathcal{D}] = -iM_a \qquad [M_a, N_b] = -2ig_{ab}\mathcal{D}$$
$$[M_a, M_b] = [N_a, N_b] = \frac{1}{2}\Gamma_{ab} = i\mathcal{L}_{ab}\ldots \tag{2.16}$$

The Hermitian generators M_a, N_a, $\mathcal{D}$, and $\mathcal{L}_{ab}$ associated to the $so(6) \sim su(4)$ algebra are given by the one-to-one correspondence

$$M_a = \frac{1}{2}\Gamma_a(1 - i\Gamma_5) \leftrightarrow -\Sigma_{a5} \qquad N_a = \frac{1}{2}\Gamma_a(1 + i\Gamma_5) \leftrightarrow \Sigma_{a6}$$
$$\mathcal{D} = \frac{1}{2}\Gamma_5 \leftrightarrow \Sigma_{56} \qquad \mathcal{L}_{ab} = -\frac{i}{2}\Gamma_{ab} \leftrightarrow \Sigma_{ab} \tag{2.17}$$

The $so(6)$ Lie algebra in 6D associated to the Hermitian generators Σ_{AB} ($A, B = 1, 2, \ldots, 6$) is defined by the commutators

$$[\Sigma_{AB}, \Sigma_{CD}] = i(g_{BC}\Sigma_{AD} - g_{AC}\Sigma_{BD} - g_{BD}\Sigma_{AC} + g_{AD}\Sigma_{BC}) \tag{2.18}$$

where g_{AB} is a diagonal 6D metric with signature (–, –, –, –, –, –). One can verify that realization (2.15) and correspondence (2.17) are consistent with the $so(6) \sim su(4)$ commutation relations (2.18). The extra $U(1)$ Abelian generator in $U(4) = U(1) \times SU(4)$ is associated with the unit $\mathbf{1}$ generator.

Because $su(4) \sim so(6)$ (isomorphic algebras) and the unitary algebra $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$, the Hermitian $u(1) \oplus so(6)$ valued field $\mathbf{A}_\mu$ may be expanded in a $Cl(3, 1, R)$ basis of Hermitian generators as

$$\mathbf{A}_\mu = a_\mu\mathbf{1} + b_\mu\Gamma_5 + e^a_\mu\Gamma_a + \mathbf{i}f^a_\mu\Gamma_a\Gamma_5 + \mathbf{i}\frac{1}{4}\omega^{ab}_\mu\Gamma_{ab} = a_\mu\mathbf{1} + A^{56}_\mu\Sigma_{56}$$
$$+ A^{a5}_\mu\Sigma_{a5} + A^{a6}_\mu\Sigma_{a6} + \frac{1}{4}A^{ab}_\mu\Sigma_{ab} \tag{2.19}$$

One should note the key presence of $\mathbf{i}$ factors in the last two (Hermitian) terms of the first line of (2.19), compared to the last two terms of (2.4) devoid of $\mathbf{i}$ factors. All the terms in (2.4) are devoid of $\mathbf{i}$ factors such that the last two terms of (2.4) are comprised of anti-Hermitian generators while the first three terms involve Hermitian generators. The conversion between the real-valued fields in the first and second lines of (2.19) is given by

$$a_\mu = a_\mu \qquad b_\mu = A^{56}_\mu \qquad A^{a5}_\mu = e^a_\mu - f^a_\nu$$
$$A^{a6}_\mu = e^a_\mu + f^a_\nu \qquad A^{ab}_\mu = \omega^{ab}_\mu \tag{2.20}$$

the conversion (2.20) is inferred from the relation

$$e^a_\mu\Gamma_a + \mathbf{i}f^a_\mu\Gamma_a\Gamma_5 = A^{a5}_\mu\Sigma_{a5} + A^{a6}_\mu\Sigma_{a6} \tag{2.21}$$

and from (2.15) (all terms in (2.21) are composed of Hermitian generators as they should). The evaluation of the $u(1) \oplus so(6)$ valued field strengths $F_{\mu\nu}$, $F^{MN}_{\mu\nu}$ $M, N = 1, 2, 3, \ldots, 6$ proceeds in a similar fashion as in the conformal gravity – Maxwell case based on the pseudo-unitary algebra $u(2, 2) = u(1) \oplus su(2, 2) \sim u(1) \oplus so(4, 2)$.

2.3. $U(p, q)$ from $U(p + q)$ via the Weyl unitary trick

In general, the unitary *compact* group $U(p + q; C)$ is related to the *noncompact* unitary group $U(p, q; C)$ by the Weyl unitary trick [33] mapping the anti-Hermitian generators of the compact group $U(p + q; C)$ to the anti-Hermitian *and* Hermitian generators of the noncompact group $U(p, q; C)$ as follows: the $(p + q) \times (p + q)$ $U(p + q; C)$ complex matrix generator is comprised of the diagonal blocks of $p \times p$ and $q \times q$ complex anti-Hermitian matrices $\mathbf{M}^\dagger_{11} = -\mathbf{M}_{11}$; $\mathbf{M}^\dagger_{22} = -\mathbf{M}_{22}$, respectively. The off-diagonal blocks are comprised of the $q \times p$ complex matrix $\mathbf{M}_{12}$ and the $p \times q$ complex matrix $-\mathbf{M}^\dagger_{12}$, that is, the off-diagonal blocks are the anti-Hermitian complex conjugates of each other. In this fashion the $(p + q) \times (p + q)$ $U(p + q; C)$ complex matrix generator $\mathbf{M}$ is anti-Hemitian $\mathbf{M}^\dagger = -\mathbf{M}$ such that upon an exponentiation $\mathbf{U}(t) = e^{t\mathbf{M}}$ it generates a unitary group element obeying the condition $\mathbf{U}^\dagger(t) = \mathbf{U}^{-1}(t)$ for t = real. This is what occurs in the $U(4)$ case.

To retrieve the noncompact group $U(2, 2; C)$ case, the Weyl unitary trick requires leaving $\mathbf{M}_{11}$, $\mathbf{M}_{22}$ intact but performing a Wick rotation of the off-diagonal block matrices $i\mathbf{M}_{12}$ and $-i\mathbf{M}^\dagger_{12}$. In this fashion, $\mathbf{M}_{11}$ and $\mathbf{M}_{22}$ still retain their anti-Hermitian character, while the off-diagonal blocks are now *Hermitian* complex conjugates of each other. This is precisely what occurs in the realization of the conformal group generators in terms of the $Cl(3, 1, R)$ algebra generators. For example, P_a, K_a both contain Hermitian Γ_a and anti-Hermitian $\Gamma_a\Gamma_5$ generators. Despite the name "unitary" group $U(2, 2; C)$, the exponentiation of the P_a and K_a generators does not furnish a truly unitary matrix obeying $\mathbf{U}^\dagger = \mathbf{U}^{-1}$. For this reason the groups $U(p, q; C)$ are more properly called *pseudo*-unitary. The complex extension of $U(p + q, C)$ is $GL(p + q; C)$. Because the algebras

$u(p+q; C)$, $u(p, q; C)$ differ only by the Weyl unitary trick, they both have identical complex extensions $gl(p+q; C)$ [33]. $gl(N, C)$ has $2N^2$ generators whereas $u(N, C)$ has N^2.

The covering of the general linear group $GL(N, R)$ admits *infinite*-dimensional spinorial representations but *not* finite-dimensional ones. For a thorough discussion of the physics of infinite-component fields and the perturbative renormalization property of metric affine theories of gravity based on (the covering of) $GL(4, R)$ we refer to [34]. The group $U(2, 2)$ consists of the 4×4 complex matrices, which preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4:

$$\langle u, u\rangle = \bar{u}^\alpha g_{\alpha\beta} u^\beta = \bar{u}^1 u^1 + \bar{u}^2 u^2 - \bar{u}^3 u^3 - \bar{u}^4 u^4 \tag{2.22}$$

obeying the *sesquilinear* conditions

$$\langle \lambda v, u\rangle = \bar{\lambda}\langle v, u\rangle \qquad \langle v, \lambda u\rangle = \lambda\langle v, u\rangle \tag{2.23}$$

where λ is a complex parameter and the bar operation denotes complex conjugation. The metric $g_{\alpha\beta}$ can be chosen to be given precisely by the chirality $(\Gamma_5)_{\alpha\beta}$ 4×4 matrix representation whose entries are $\mathbf{1}_{2\times2}$, $-\mathbf{1}_{2\times2}$ along the main diagonal blocks, respectively, and 0 along the off-diagonal blocks. The Lie algebra $su(2, 2) \sim so(4, 2)$ corresponds to the conformal group in 4D. The special unitary group $SU(p+q; C)$ in addition to being sesquilinear metric-preserving is also volume-preserving.

The group $U(4)$ consists of the 4×4 complex matrices that preserve the *sesquilinear* symmetric metric $g_{\alpha\beta}$ associated to the following quadratic form in C^4:

$$\langle u, u\rangle = \bar{u}^\alpha g_{\alpha\beta} u^\beta = \bar{u}^1 u^1 + \bar{u}^2 u^2 + \bar{u}^3 u^3 + \bar{u}^4 u^4 \tag{2.24}$$

The metric $g_{\alpha\beta}$ is now chosen to be given by the unit $\mathbf{1}_{\alpha\beta}$ diagonal 4×4 matrix. The $U(4) = U(1) \times SU(4)$ metric-preserving group transformations are generated by the 15 Hermitian generators Σ_{AB} and the unit $\mathbf{1}$ generator.

In the most general case one has the following isomorphisms of Lie algebras [33]:

$$\begin{aligned} &so(5,1) \sim su^*(4) \sim sl(2, H) \qquad so^*(6) \sim su(3,1) \\ &so(3,2) \sim sp(4, R) \qquad so(4,2) \sim su(2,2) \\ &so(3,3) \sim sl(4, R) \qquad so(6) \sim su(4), \ldots \end{aligned} \tag{2.25}$$

where the asterisks like $su^*(4)$ and $so^*(6)$ denote the algebras associated with the *noncompact* versions of the compact groups $SU(4)$, $SO(6)$. $sl(2, H)$ is the special linear Mobius algebra over the field of quaternions H. The $SU(4)$ group is a two-fold covering of $SO(6)$ but their algebras are isomorphic.

2.4. Complex conformal gravity and $U(4) \times U(4)$ Yang–Mills from $Cl(5, C)$

To complete this section it is necessary to recall the following isomorphisms among real and complex Clifford algebras [16]

$$\begin{aligned} &Cl(2m+1, C) = Cl(2m, C) \oplus Cl(2m, C) \sim M(2^m, C) \oplus M(2^m, C) \\ &\Rightarrow Cl(5, C) = Cl(4, C) \oplus Cl(4, C) \end{aligned} \tag{2.26a}$$

and

$$Cl(4, C) \sim M(4, C) \sim Cl(4, 1, R) \sim Cl(2, 3, R) \sim Cl(0, 5, R) \tag{2.26b}$$

$$Cl(4, C) \sim M(4, C) \sim Cl(3, 1, R) \oplus \mathbf{i}Cl(3, 1, R) \sim M(4, R) \oplus \mathbf{i}M(4, R) \tag{2.26c}$$

$$Cl(4, C) \sim M(4, C) \sim Cl(2, 2, R) \oplus \mathbf{i}Cl(2, 2, R) \sim M(4, R) \oplus \mathbf{i}M(4, R) \tag{2.26d}$$

where $M(4, R)$, $M(4, C)$ is the 4×4 matrix algebra over the reals and complex numbers, respectively. From each one of the $Cl(3, 1, R)$ algebra factors in (2.26c) of the complex $Cl(4, C)$ algebra, one can generate a $u(2, 2)$ algebra by writing the $u(2, 2)$ generators explicitly in terms of the $Cl(3, 1, R)$ gamma matrices as displayed in (2.7); that is, one may convert a $Cl(3, 1, R)$ gauge theory into a onformal gravity – Maxwell theory based on $U(2, 2) = SU(2, 2) \times U(1)$. Therefore, a $Cl(4, C)$ gauge theory is algebraically equivalent to a *bi*-conformal gravity – Maxwell theory based on the complex group $U(2, 2) \otimes \mathbf{C} = GL(4, C)$; that is, the $Cl(4, C)$ gauge theory is algebraically equivalent to a *complexified* conformal gravity – Maxwell theory in four real dimensions based on the complex algebra $u(2, 2) \oplus \mathbf{i}u(2, 2) = gl(4, C)$. The algebra $gl(N, C)$ is the complex extension of $u(p, q)$ for all p, q such that $p + q = N$.

Furthermore, from each $Cl(3, 1, R)$ commuting subalgebra inside the $Cl(4, C)$ algebra one can also generate a $u(4) = u(1) \oplus su(4) \sim u(1) \oplus so(6)$ algebra by writing the latter generators in terms of the $Cl(3, 1, R)$ gamma matrices as displayed explicitly in (2.15). Therefore, the $Cl(4, C)$ gauge theory is also algebraically equivalent to a Yang–Mills gauge theory based on the algebra $u(4) \oplus \mathbf{i}u(4) = gl(4, C)$ and associated with the *two* $Cl(3, 1, R)$ commuting subalgebras inside $Cl(4, C)$. The complex group is $U(4) \otimes \mathbf{C} = GL(4, C)$ also.

From (2.26d), $Cl(4, C) \sim Cl(4, 1, R)$ one learns that the complex Clifford $Cl(4, C)$ algebra is also isomorphic to a *real* Clifford algebra $Cl(4, 1, R)$ (and also to $Cl(2, 3, R)$, $Cl(0, 5, R)$). A Wick rotation (Weyl unitary trick) transforms $Cl(4, 1, R) \to Cl(3, 2, R) = Cl(3, 1, R) \sim M(4, R) \oplus M(4, R)$ such that there are two commuting subalgebras of $Cl(3, 2, R)$, which are isomorphic to $Cl(3, 1, R)$.

From each one of the latter $Cl(3, 1, R)$ algebras one can build an $u(4)$ (and $u(2, 2)$) algebra as described earlier. A typical example of this feature in ordinary Lie algebras is the case of $so(3) \sim su(2)$ such that there are two commuting subalgebras of $so(4)$ and isomorphic to $so(3)$ furnishing the decomposition $so(4) = su(2) \oplus su(2) \sim so(3) \oplus so(3)$. Concluding, one can generate a $U(4) \times U(4)$ Yang–Mills gauge theory from a $Cl(4, C)$ gauge theory via a $Cl(4, 1, R)$ gauge theory (based on a *real* Clifford algebra) after the Wick rotation (Weyl unitary trick) procedure to the $Cl(3, 2, R)$ algebra is performed.

The physical reason that one needs a $U(4) \times U(4)$ Yang–Mills theory is because the group $U(4)$ by itself is *not* large enough to accommodate the Standard Model group $SU(3) \times SU(2) \times U(1)$ as its maximally compact subgroup [35]. The GUT groups $SU(5)$ and $SU(2) \times SU(2) \times SU(4)$ are large enough to achieve this goal. In general, the group $SU(m+n)$ has $SU(m) \times SU(n) \times U(1)$ for compact subgroups. Therefore, $SU(4) \to SU(3) \times U(1)$ or $SU(4) \to SU(2) \times SU(2) \times U(1)$ is allowed, but one cannot have $SU(4) \to SU(3) \times SU(2)$. For this reason one cannot rely only on a $Cl(4, C) = Cl(3, 1, R) \oplus \mathbf{i}Cl(3, 1)$ gauge theory to build a unifying model; that is, because one cannot have the branching $SU(4) \to SU(3) \times SU(2)$, one would not be able to generate the full Standard Model group despite that the other group inside $Cl(4, C)$ given by $U(2, 2) = SU(2, 2) \times U(1)$ furnishes conformal gravity *and* Maxwell's electromagnetism (EM) based on $U(1)$.

A breaking [29, 36] of $U(4) \times U(4) \to SU(2)_L \times SU(2)_R \times SU(4)$ leads to the PS [37] GUT group, which contains the Standard Model group, which in turn, breaks down to the ordinary Maxwell EM $U(1)_{EM}$ and color (QCD) group $SU(3)_c$ after the following chain of symmetry breaking patterns

$$\begin{aligned} &SU(2)_L \times SU(2)_R \times SU(4) \to SU(2)_L \times U(1)_R \times U(1)_{B-L} \times SU(3)_c \\ &\to SU(2)_L \times U(1)_Y \times SU(3)_c \to U(1)_{EM} \times SU(3)_c \end{aligned} \tag{2.27}$$

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where B–L denotes the baryon minus lepton number charge; Y is the hypercharge; and the Maxwell EM charge is $Q = I_3 + (Y/2)$ where I_3 is the third component of the $SU(2)_L$ isospin. It is noteworthy that because we already identified the $U(1)_{EM}$ symmetry stemming from the ($U(2, 2)$ group-based) conformal gravity – Maxwell sector, it is not necessary to follow the symmetry-breaking pattern of the second line in (2.27) to retrieve the desired $U(1)_{EM}$ symmetry.

The upshot of the $Cl(5, C) = Cl(4, C) \oplus Cl(4, C)$ algebraic decomposition is that the group structure given by the *direct* products $[U(2, 2) \times U(2, 2)]_{\text{space–time}} \times [U(4) \times U(4)]_{\text{Yang–Mills}}$ is ultimately tied down to four dimensions. Decomposing $U(2, 2) = SU(2, 2) \times U(1)$ and focusing on the conformal group $SU(2, 2)$, we see that it does *not* violate the Coleman–Mandula theorem because the space–time symmetries (conformal group $SU(2, 2)$ in the absence of a mass gap, Poincare group when there is mass gap) do *not* mix with the internal symmetries. Similar considerations apply to the supersymmetric case when the symmetry group structure is given by the *direct* product of the superconformal group (in the absence of a mass gap) with an internal symmetry group so that the Haag–Lopuszanski–Sohnius theorem is not violated. There is an extra $U(1)$ symmetry that needs further clarification. It is likely that it can be related to a global symmetry that survives at lower energies; see the following sections.

2.5. Gravity, trinification, and PS models from *Cl*(5, *C*) gauge field theories

In ref. 16 we briefly mentioned that under the Weyl unitary "trick" one of the $U(2, 2)$ group factors becomes $U(2, 2) \to U(4)$ so that $Cl(5, C) > U(2, 2) \times [U(4) \times U(4) \times U(4)]$ resulting in a *four*-generation trinification model. The first factor group $U(2, 2) = SU(2, 2) \times U(1)$ contains the conformal group $SU(2, 2)$, it acts on the 4D space–time and does not mix with the trinification group $[U(4)]^3 = U(4) \times U(4) \times U(4)$. Meaning that the commutators of the $U(2, 2)$ generators with the $[U(4)]^3$ ones are all vanishing.

Recently in ref. 38 a conformal completion of the Standard Model with a fourth generation was advanced with predictions of new gauge bosons, bifundamental fermions and scalars accessible by the Large Hadron Collider can be found. Related to four fermion generations, the authors [39] argued the possibility that fermion masses, in particular quarks, might originate through the condensation of a fourth family that interacts with all of the quarks via a contact four-fermion term coming from the existence of torsion on the space–time. A fourth-generation model and a kinematic Higgs mechanism to construct chiral fermion masses in the Standard Model based on Dirac–Kahler fermions was presented by Jourjine [40]. The mass spectrum was computed and the electron neutrino and the fourth neutrino masses are related via a see-saw-like mechanism. The relevance of Dirac–Kahler fermions is that their description fits naturally into the polyvector decomposition of the Clifford algebra generators into scalars, vectors, bivectors, trivectors, etc.

A breaking of $U(4) \times U(4) \times U(4) \to SU(3)_C \times SU(3)_L \times SU(3)_R$ leads to the trinification gauge group proposed long ago by Glashow [41] involving three generations of fermions. The group is combined with a discrete symmetry group Z_3 exchanging left, right, and color symmetries. A breaking of $SU(3)_C \times SU(3)_L \times SU(3)_R \to SU(3)_C \times SU(2)_W \times U(1)_Y$ furnishes the Standard Model gauge group.

Within the context of string and M-theory, a $U(3)_C \times U(3)_L \times U(3)_R$ gauge symmetry from intersecting D-branes was found by Leontaris [42]. This is equivalent to the trinification model extended by three $U(1)$ factors, which survive as global symmetries in the low-energy effective model. The Standard Model fermions are accommodated in the three possible bifundamental multiplets represented by strings with endpoints attached on different brane-stacks of this particular setup.

A Dp-brane is an extended object in p-dimensions whose world volume is $p + 1$ dimensional. In D-branes model building one exploits the fact that a stack of N parallel, almost coincident D-branes gives rise to a $U(N)$ gauge group. Chirality arises when intersecting branes are wrapped on a torus with the chiral fermions sitting in the various *intersections* of the D-branes configuration. Here, the six-dimensional compact space is taken to be a 6D factorizable torus $T_6 = (T_2)^3$.

To construct the D-brane analogue of the trinification model, Leontaris [42] considered three stacks of D6-branes, each stack containing three parallel almost coincident branes giving rise to the gauge symmetry. Four stacks of four parallel almost coincident D-branes will furnish the group $U(4) \times U(4) \times U(4) \times U(4) > \mathrm{Cl}(5, C)$. The Standard Model fermions are represented by open strings attached to two different brane-stacks and belong to $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ representations as is the case of the $SU(3)^3$ trinification model. For further details we refer to ref. 42. More recently the physical Higgs mass (pole mass) was found to be 125 ± 1.4 GeV in agreement with the experimental results and based on a study of the trinification subgroup of E_6 by Stech [43].

The fermionic matter and Higgs sector of the Standard Model within the context of Clifford gauge field theories has been analyzed in ref. 16. The 16 fermions of each generation can be assembled into the entries of a 4 × 4 matrix representation of the $Cl(4)$ algebra whose 16 generators are Γ^A, A = 1, 2, 3, ..., 16. The latter generators can be represented in terms of 4 × 4 matrices $(\Gamma^A)_{ij}$ whose indices are $i, j = 1, 2, 3, 4$. A fermion field Ψ^A_α carries double indices, A represents an internal $Cl(4)$-valued gauge index, while α represents a $Cl(3, 1)$ spinor index associated with the four-dimensional space–time. The left-handed sector can be written as

$$\sum_A \Psi^A_{\alpha,\mathrm{L}}(\Gamma_A)_{ij} \equiv \begin{pmatrix} \mathrm{n_e} & \mathrm{u_r} & \mathrm{u_b} & \mathrm{u_g} \\ \mathrm{e} & \mathrm{d_r} & \mathrm{d_b} & \mathrm{d_g} \\ \mathrm{e}^+ & \bar{\mathrm{d}}^{\bar{\mathrm{r}}} & \bar{\mathrm{d}}^{\bar{\mathrm{b}}} & \bar{\mathrm{d}}^{\bar{\mathrm{g}}} \\ \bar{\mathrm{n}}_{\mathrm{e}} & \bar{\mathrm{u}}^{\bar{\mathrm{r}}} & \bar{\mathrm{u}}^{\bar{\mathrm{b}}} & \bar{\mathrm{u}}^{\bar{\mathrm{g}}} \end{pmatrix}_{\mathrm{L}} \tag{2.28a}$$

the right-handed sector is

$$\sum_A \Psi^A_{\alpha,\mathrm{R}}(\Gamma_A)_{ij} \equiv \begin{pmatrix} \mathrm{n_e} & \mathrm{u_r} & \mathrm{u_b} & \mathrm{u_g} \\ \mathrm{e} & \mathrm{d_r} & \mathrm{d_b} & \mathrm{d_g} \\ \mathrm{e}^+ & \bar{\mathrm{d}}^{\bar{\mathrm{r}}} & \bar{\mathrm{d}}^{\bar{\mathrm{b}}} & \bar{\mathrm{d}}^{\bar{\mathrm{g}}} \\ \bar{\mathrm{n}}_{\mathrm{e}} & \bar{\mathrm{u}}^{\bar{\mathrm{r}}} & \bar{\mathrm{u}}^{\bar{\mathrm{b}}} & \bar{\mathrm{u}}^{\bar{\mathrm{g}}} \end{pmatrix}_{\mathrm{R}} \tag{2.28b}$$

We have arranged the entries of these 4 × 4 matrices to accommodate the chiral fermions into representations of the PS $SU(4) \times SU(2)_L \times SU(2)_R$ group such that the preceding 4 × 4 matrix entries admit the following $SU(4) \times SU(2)_L \times SU(2)_R$ decomposition. The left-handed fermions are displayed in the following representation of the PS group:

$$(\mathbf{4}, \mathbf{2}, \mathbf{1}) = \begin{pmatrix} \nu_{\mathrm{e}} & \mathrm{u_r} & \mathrm{u_b} & \mathrm{u_g} \\ \mathrm{e} & \mathrm{d_r} & \mathrm{d_b} & \mathrm{d_g} \end{pmatrix}_{\mathrm{L}} \tag{2.29a}$$

Because the right-handed antiparticles feel the left-handed weak $SU(2)_L$ force [35] one has

$$(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}) = \begin{pmatrix} \mathrm{e}^+ & \bar{\mathrm{d}}^{\bar{\mathrm{r}}} & \bar{\mathrm{d}}^{\bar{\mathrm{b}}} & \bar{\mathrm{d}}^{\bar{\mathrm{g}}} \\ \bar{\nu}_{\mathrm{e}} & \bar{\mathrm{u}}^{\bar{\mathrm{r}}} & \bar{\mathrm{u}}^{\bar{\mathrm{b}}} & \bar{\mathrm{u}}^{\bar{\mathrm{g}}} \end{pmatrix}_{\mathrm{R}} \tag{2.29b}$$

Because the left-handed antiparticles feel the right-handed weak $SU(2)_R$ force [35] one has

$$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) = \begin{pmatrix} \mathrm{e}^+ & \bar{\mathrm{d}}^{\bar{\mathrm{r}}} & \bar{\mathrm{d}}^{\bar{\mathrm{b}}} & \bar{\mathrm{d}}^{\bar{\mathrm{g}}} \\ \bar{\nu}_{\mathrm{e}} & \bar{\mathrm{u}}^{\bar{\mathrm{r}}} & \bar{\mathrm{u}}^{\bar{\mathrm{b}}} & \bar{\mathrm{u}}^{\bar{\mathrm{g}}} \end{pmatrix}_{\mathrm{L}} \tag{2.29c}$$

and, finally, the right-handed fermions are displayed in the representation

$$(\mathbf{4},\mathbf{1},\mathbf{2}) = \begin{pmatrix} \mathrm{n_e} & \mathrm{u_r} & \mathrm{u_b} & \mathrm{u_g} \\ \mathrm{e} & \mathrm{d_r} & \mathrm{d_b} & \mathrm{d_g} \end{pmatrix}_{\mathrm{R}} \tag{2.29d}$$

where we have omitted the space–time spinorial indices $\alpha = 1, 2, 3, 4$ in each one of the entries of these matrices. In particular, e, ν_e denote the electron and its neutrino. The subscripts r, b, g denote the red, blue, and green color of the up and down quarks, u, d. The subscripts $\bar{\mathrm{r}}$, $\bar{\mathrm{b}}$, $\bar{\mathrm{g}}$ denote the anti-red, anti-blue, and anti-green color of the up and down antiquarks, $\bar{\mathrm{u}}$, $\bar{\mathrm{d}}$. The antiparticles are denoted by $\bar{\mathrm{e}}$, $\bar{\nu}_e$, $\bar{\mathrm{u}}$, $\bar{\mathrm{d}}$. The remaining chiral fermions (Weyl spinors) of the second and third generation have identical decomposition as the one displayed in (2.28) and (2.29). One simply replaces e with μ and τ for the muon and tau particles, respectively; the neutrino ν_e for the neutrinos ν_μ, ν_τ, and the u, d quarks for the charm, strange c, s and top, bottom t, b quarks, respectively.

The algebra of GUTs, related to the $SO(10)$, $SU(5)$ and PS group was analyzed from a different perspective than the Clifford algebraic one presented here [35]. The upshot of having the $Cl(4)$-algebraic description of the 16 left- and right-handed fermions (Weyl spinors) in (2.28) is that it is consistent with the $SU(4)$ color symmetry (force) of the PS model. The leptons are seen as the carriers of the white "fourth" color. Furthermore, one is confined to the observed four space–time dimensions.

In general, the fermionic matter kinetic terms for n_f generations is

$$\mathcal{L}_m = \sum_{i=1}^{n_f} \overline{\Psi}^A_{\alpha i} \Gamma^\mu_{\alpha\beta} \left(\delta_{AC}\partial_\mu + f_{ABC}\mathcal{A}^B_\mu\right)\Psi^C_{\beta i} \tag{2.30}$$

where the indices $i = 1, 2, 3, \ldots, n_f$ extend over the number of generations (flavors) and $A, B, C = 1, 2, 3, \ldots, 16$. f_{ABC} denote the structure constants of the $Cl(4)$ gauge algebra.

Because the PS $SU(4) \times SU(2)_L \times SU(2)_R$ group arises from the symmetry breaking of *one* of the $SU(4)$ factors in $SU(4) \times SU(4) \times SU(4)$, and given by $SU(4) \to SU(2)_L \times SU(2)_R \times U(1)_Z$, this requires taking the following vacuum expectation value (VEV) of the Higgs scalar:

$$\langle\Phi\rangle \equiv v_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.31}$$

Taking the VEV of the other Higgs scalar

$$\langle\tilde{\Phi}\rangle \equiv v_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{2.32}$$

leads to a breaking of $SU(4) \to SU(3)_c \times U(1)_{B-L}$. Therefore, an overall breaking of $SU(4) \times SU(4)$ contains the PS model in the intermediate stage as follows:

$$\begin{aligned} &SU(4) \times SU(4) \to [SU(4) \times SU(2)_L \times SU(2)_R]_{PS} \times U(1)_Z \\ &\to SU(3)_c \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R \times U(1)_Z \end{aligned} \tag{2.33}$$

The Higgs potential $V(\Phi, \tilde{\Phi})$ involving quadratic and quartic powers of the fields is of the form

$$\begin{aligned} V = &-m_1^2\mathrm{Tr}(\Phi^2) + \lambda_1[\mathrm{Tr}(\Phi^2)]^2 + \lambda_2\mathrm{Tr}(\Phi^4) - m_2^2\mathrm{Tr}(\tilde{\Phi}^2) \\ &+ \lambda_3[\mathrm{Tr}(\tilde{\Phi}^2)]^2 + \lambda_4\mathrm{Tr}(\tilde{\Phi}^4) + \lambda_5\mathrm{Tr}(\Phi^2\tilde{\Phi}^2) + \lambda_6\mathrm{Tr}(\Phi\tilde{\Phi}\Phi\tilde{\Phi}) \end{aligned} \tag{2.34}$$

A further symmetry breaking

$$U(1)_{B-L} \times SU(2)_R \times U(1)_Z \to U(1)_Y \tag{2.35}$$

requires additional Higgs fields leading to the Standard Model

$$SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{EM} \tag{2.36}$$

For further details of the Yukawa coupling terms furnishing masses for the quarks and leptons we refer to ref. 16. In C-space the couplings are of the form $f_{ABC}\overline{\Psi}^A\mathcal{A}^B_0\Psi^C$, $f_{ABC}\overline{\Psi}^A\mathcal{A}^B_5\Psi^C$ (after taking the VEV of the Higgs scalars) associated to the C-space fermionic kinetic terms $\overline{\Psi}_A\Gamma^M(D_M)^{AB}\Psi_B$ [16] because the Higgs scalar fields in C-space are identified with the scalar and pseudo-scalar components of the C-space $Cl(4)$-valued gauge field $\Phi^A = \mathcal{A}^A_0$ and $\epsilon_{\mu\nu\rho\tau}\tilde{\Phi}^A = \mathcal{A}^A_{\mu\nu\rho\tau}$ as shown in ref. 16. The kinetic terms for the Higgs field $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ are contained in the field strength components $F_{0M}F^{0M}$ associated to the $F_{MN}F^{MN}$ terms where M, N are polyvector-valued indices corresponding to the coordinates of the 16-dimensional ($2^4 = 16$) C-space associated with four space–time dimensions. The 0 index corresponds to the unit (scalar) element of the space–time Clifford algebra $Cl(3, 1)$. The five index corresponds to the $\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4$ (pseudo-scalar) element of the space–time Clifford algebra $Cl(3, 1)$.

Whereas, the kinetic terms for the other Higgs field $(D_\mu\tilde{\Phi})^\dagger(D^\mu\tilde{\Phi})$ are contained in the components $F_{5M}F^{5M}$ associated to the $F_{MN}F^{MN}$ terms. Inserting the VEV of the Higgs scalars into their kinetic terms, after redefining the fields such that the new fields have zero VEV, yields the mass terms from the gauge fields associated to the broken gauge symmetries.

There is another symmetry-breaking branch that leads to the Standard Model and which does not contain the PS model. This requires breaking one of the $SU(4)$ factors as

$$SU(4) \times SU(4) \to SU(3)_c \times SU(4) \times U(1)_{B-L} \tag{2.37}$$

leading to a partial unification model based on $SU(4) \times U(1)_{B-L}$, which can be broken down to the minimal left–right model via the Higgs mechanism [36]. More work remains to be done to verify whether or not this approach to unification is feasible. In particular, a thorough analysis of the parameters involved in the potential $V(\Phi, \tilde{\Phi})$, the gauge couplings g, the expectation values parameters v_1, v_2, ..., is warranted.

A unified model of strong, weak, and EM interactions based on the flavor–color group $SU(4)_f \times SU(4)_c$ of PS has been described by Rajpoot and Singer [37]. Fermions were placed in left–right multiplets, which transform as the representation $(\bar{4}, 4)$ of $SU(4)_f \times SU(4)_c$. Further investigation is warranted to explore the group $SU(4)_f \times SU(4)_c$ of PS within the context of the $U(4) \times U(4)$ group symmetry associated with the $Cl(4, C)$ algebra presented here.

2.6. Embedding $U(4)$ into $SO(8) \subset Cl(8)$, fermionic oscillator algebras, and $SO(10)$ GUT

The $u(4)$ algebra can also be realized in terms of $so(8)$ generators, and in general, $u(N)$ algebras admit realizations in terms of $so(2N)$ generators [29]. Given the Weyl–Heisenberg "superalgebra" involving the N fermionic creation and annihilation (oscillators) operators

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = 0,\ \{a_i^\dagger, a_j^\dagger\} = 0 \qquad i, j = 1, 2, 3, \ldots, N \tag{2.38}$$

one can find a realization of the $u(N)$ algebra bilinear in the oscillators as $E_i^j = a_i^\dagger a_j$ and such that the commutators

$$\left[E_i^j, E_k^l\right] = a_i^\dagger a_j a_k^\dagger a_l - a_k^\dagger a_l a_i^\dagger a_j = a_i^\dagger\left(\delta_{jk} - a_k^\dagger a_j\right)a_l - a_k^\dagger\left(\delta_{li} - a_i^\dagger a_l\right)a_j = a_i^\dagger(\delta_{jk})a_l - a_k^\dagger(\delta_{li})a_j = \delta_k^j E_i^l - \delta_i^l E_k^j \quad (2.39)$$

reproduce the commutators of the Lie algebra $u(N)$ because

$$-a_i^\dagger a_k^\dagger a_j a_l + a_k^\dagger a_i^\dagger a_l a_j = -a_k^\dagger a_i^\dagger a_l a_j + a_k^\dagger a_i^\dagger a_l a_j = 0 \quad (2.40)$$

because the anticommutation relations, (2.38), yield a double negative sign (–)(–) = + in (2.40). Furthermore, one also has an explicit realization of the Clifford algebra $Cl(2N)$ Hermitian generators by defining the even-number and odd-number generators as

$$\Gamma_{2j} = \frac{1}{2}\left(a_j + a_j^\dagger\right) \qquad \Gamma_{2j-1} = \frac{1}{2i}\left(a_j - a_j^\dagger\right) \quad (2.41)$$

The Hermitian generators of the $so(2N)$ algebra are defined as usual $\Sigma_{mn} = (i/4)[\Gamma_m, \Gamma_n]$ where $m, n = 1, 2, ..., 2N$. Therefore, the $u(4)$, $so(8)$, and $Cl(8)$ algebras admit an explicit realization in terms of the fermionic Weyl–Heisenberg oscillators a_i, $a_j^\dagger$ for $i, j = 1, 2, 3, 4$. $u(4)$ is a subalgebra of $so(8)$, which in turn is a subalgebra of the $Cl(8)$ algebra. The conformal algebra in 8D is $so(8, 2)$ and also admits an explicit realization in terms of the $Cl(8)$ generators, similar to the realization of the algebra $so(4, 2) \sim su(2, 2)$ in terms of the $Cl(3, 1, R)$ generators as displayed in (2.7). The compact version of the group $SO(8, 2)$ is $SO(10)$, which is a GUT group candidate. In particular, the algebras $u(5)$, $so(10)$, and $Cl(10)$ admit a realization in terms of the fermionic Weyl–Heisenberg oscillators a_i, $a_j^\dagger$ for $i, j = 1, 2, 3, 4, 5$.

3. 4D gravity, *SU*(3) × *SU*(2) × *U*(1) Yang–Mills from 8D quaternionic gravity

In this section we review how gravity and $SU(3) \times SU(2) \times U(1)$ Yang–Mills in 4D can be obtained from 8D quaternionic gravity after a Kaluza–Klein compactification along the internal four-dimensional space [17].

It has been argued by Batakis [13] that a Kaluza–Klein compactification of 8D gravity on CP^2 involving a nontrivial *torsion* may bypass the no-go theorems by Witten that one cannot obtain the group $SU(3) \times SU(2) \times U(1)$ from a Kaluza–Klein mechanism in 8D. It was *assumed* by Batakis [13] that *if* the torsion components, $T^a_{\mu\nu}$, were proportional to $F^I_{\mu\nu}e^a_I$, where e^a_I is a vielbein employed to change the $SU(2) \times U(1)$ group index $I = 1, 2, 3, 4$ to the internal 4D space CP^2 index $a = 1, 2, 3, 4$, the 8D Lagrangian corresponding to the curvature scalar and associated with a connection with contorsion K: $\mathbf{R}(\Gamma + K) = R(\Gamma) + (K)^2 + \nabla K$ yields a gravitational and $SU(3) \times SU(2) \times U(1)$ Yang–Mills theory upon compactification on $CP^2 = SU(3)/SU(2) \times U(1)$. The *problem* was that *no* proof was presented in ref. 13 that shows why $T^a_{\mu\nu}$ is proportional to $F^I_{\mu\nu}e^a_I$.

For these reasons, in this section we shall build an unification model of 4D gravity and $SU(3) \times SU(2) \times U(1)$ Yang–Mills theory (in the absence of matter) obtained from a Kaluza–Klein compactification of 8D quaternionic gravity on CP^2, rather than introducing by hand the torsion squared terms [13]. In this way we avoid the problems encountered by Einstein and co-workers [44, 45], and also construct unified theories that contain the electroweak force and gravity in 4D. Our results differ also from the construction in ref. 46 to unify the electroweak force with gravity in 4D after complexifying the de Sitter group.

A geometrical treatment of a non-Riemannian geometry including an internal complex, quaternionic and octonionic space has been investigated by several authors [44, 45, 47, 48]. A quaternionic-valued metric is defined as

$$\mathbf{g}_{\mu\nu} = g_{\mu\nu}\mathbf{e}_o + \mathbf{g}^i_{[\mu\nu]}\mathbf{e}_i \qquad \mathbf{e}_i\mathbf{e}_j = -\delta_{ij}\mathbf{e}_o + \epsilon_{ijk}\mathbf{e}_k \qquad i, j, k = 1, 2, 3 \quad (3.1)$$

obeying the symmetry condition $\mathbf{g}^\dagger_{\mu\nu} = \mathbf{g}_{\nu\mu}$ where the Hermitian conjugation is taken in the internal quaternionic space. Namely, one can represent the generators of the quaternionic algebra in terms of the Hermitian Pauli spin 2 × 2 matrices $\boldsymbol{\sigma}_i$ and the unit 2 × 2 matrix as $\mathbf{e}_o = \mathbf{1}_{2\times 2}$; $\mathbf{e}_i = -i\boldsymbol{\sigma}_i$. Hence the Hermitian conjugation is carried on the 2 × 2 matrices. The physical distance is

$$ds^2 = \frac{1}{2}\mathrm{Tr}\left(\mathbf{g}_{\mu\nu}dx^\mu dx^\nu\right) = g_{(\mu\nu)}dx^\mu dx^\nu \quad (3.2)$$

due to the traceless condition of the Pauli spin matrices and commuting nature of the coordinates. One may choose $g_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]}$ and maintain the Hermiticity condition $\mathbf{g}^\dagger_{\mu\nu} = \mathbf{g}_{\nu\mu}$ if $(ig_{[\mu\nu]}\mathbf{e}_o)^\dagger = -ig_{[\mu\nu]}\mathbf{e}_o$; that is, if one includes a complex conjugation on i as well, which is compatible with the fact that $(\mathbf{e}_i)^\dagger = (-i\boldsymbol{\sigma}_i)^\dagger = +i\boldsymbol{\sigma}_i = -\mathbf{e}_i$ because the Pauli spin 2 × 2 matrices $\boldsymbol{\sigma}_i$ are taken to be Hermitian.

The quaternionic-valued connection is

$$\Upsilon^\sigma_{\mu\rho} = \left(\Gamma^\sigma_{(\mu\rho)} + i\Gamma^\sigma_{[\mu\rho]}\right)e_o + \left(\Theta^\sigma_{[\mu\rho]}\right)^i e_i \quad (3.3)$$

we explicitly write $(\mu\rho)$, $[\mu\rho]$ to denote the symmetry and antisymmetry properties, respectively, of the connection components. We will show how a Kaluza–Klein compactification in the internal space CP^2, from 8D to 4D, yields a gravitational, $SU(3) \times SU(2) \times U(1)$ Yang–Mills theory in 4D.

The gravitational and $U(1)$ Maxwell's EM sector are encoded, respectively, in the symmetric piece $\Gamma^\sigma_{(\mu\rho)}e_o$ and antisymmetric piece $i\Gamma^\sigma_{[\mu\rho]}e_o$ corresponding to the unit element $\mathbf{e}_o$ of the quaternionic-algebra-valued connection. The $SU(2)$ sector is encoded in the internal part $(\Theta^\sigma_{[\mu\rho]})^i e_i$. The $SU(3)$ Yang–Mills sector arises upon the Kaluza–Klein compactification resulting from the isometry group of the CP^2 internal space. Therefore, from a pure quaternionic gravity in 8D one can obtain a grand unified field theory of gravity and the standard model group $SU(3) \times SU(2) \times U(1)$ in 4D.

This result can be attained by restricting $\Gamma^\sigma_{[\mu\rho]} = \delta^\sigma_\rho A_\mu - \delta^\sigma_\mu A_\rho$ to be the Einstein–Schrodinger connection, where A_μ is the EM field. Due to the antisymmetry, $\Gamma^\sigma_{[\mu\rho]}$ transforms as a tensor. This is not the case with $\Gamma^\sigma_{(\mu\rho)}$. The internal part of the connection $\Theta^\sigma_{[\mu\rho]}$ is restricted to be of the form $(\delta^\sigma_\rho\Theta^i_\mu - \delta^\sigma_\mu\Theta^i_\rho)e_i$, $i = 1, 2, 3$, such that the commutator becomes $[\Theta_\mu, \Theta_\nu] = 2\Theta^i_\mu\Theta^j_\nu\epsilon_{ijk}e_k$. The quaternionic-valued curvature

$$\mathbf{R}^\sigma_{\mu\nu\rho} = \partial_\mu\Upsilon^\sigma_{\nu\rho} - \partial_\nu\Upsilon^\sigma_{\mu\rho} + \Upsilon^\sigma_{\mu\tau}\Upsilon^\tau_{\nu\rho} - \Upsilon^\sigma_{\nu\tau}\Upsilon^\tau_{\mu\rho} = \left(R^\sigma_{\mu\nu\rho} + iF^\sigma_{\mu\nu\rho}\right)e_o + \left(\mathbf{P}^\sigma_{\mu\nu\rho}\right)^k e_k + \text{extra terms} \quad (3.4)$$

has for components the following terms: the standard Riemannian curvature tensor written in terms of the Christoffel symbols as

$$R^\sigma_{\mu\nu\rho} = \partial_\mu\Gamma^\sigma_{(\nu\rho)} - \partial_\nu\Gamma^\sigma_{(\mu\rho)} + \Gamma^\sigma_{(\mu\tau)}\Gamma^\tau_{(\nu\rho)} - \Gamma^\sigma_{(\nu\tau)}\Gamma^\tau_{(\mu\rho)} \quad (3.5)$$

The tensor containing the Maxwell field strength is

$$F^\sigma_{\mu\nu\rho} = \delta^\sigma_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu) + \delta^\sigma_\mu\partial_\nu A_\rho - \delta^\sigma_\nu\partial_\mu A_\rho \quad (3.6)$$

such that the contraction $F^\sigma_{\mu\nu\sigma} = (D-1)F_{\mu\nu}$ in D dimensions is proportional to the $U(1)$ EM field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. And, finally, the $SU(2)$ field strength is encoded in the internal part of the curvature tensor, which can be written as

$$\mathbf{P}_{\mu\nu} = \partial_\mu\Theta_\nu - \partial_\nu\Theta_\mu + [\Theta_\mu, \Theta_\nu] = \left(\partial_\mu\Theta_\nu^k - \partial_\nu\Theta_\mu^k\right)e_k + 2\Theta_\mu^i\Theta_\nu^j\epsilon_{ijk}e_k \tag{3.7}$$

leading to

$$\mathbf{P}^\sigma_{\mu\nu\rho} = \left(\mathbf{P}^\sigma_{\mu\nu\rho}\right)^k e_k = \delta^\sigma_\rho(\mathbf{P}_{\mu\nu})^k e_k = \delta^\sigma_\rho(\partial_\mu\Theta_\nu - \partial_\nu\Theta_\mu + [\Theta_\mu, \Theta_\nu])^k e_k \tag{3.8}$$

There are *extra* terms in (2.4) involving products of the form

$$\Gamma^\sigma_{(\mu\tau)}\Gamma^\tau_{[\nu\rho]} \qquad \Gamma^\sigma_{(\mu\tau)}\left(\Theta^\tau_{[\nu\rho]}\right)^k \qquad \Gamma^\sigma_{[\mu\tau]}\Gamma^\tau_{[\nu\rho]} \qquad \Gamma^\sigma_{[\mu\tau]}\left(\Theta^\tau_{[\nu\rho]}\right)^k \tag{3.9}$$

and for simplicity are not written down. The first two terms in (3.9) can be reabsorbed inside the ordinary derivatives to yield "covariantized" $SU(2) \times U(1)$ field strengths involving the *analog* of covariant-like derivatives ∇_μ acting on the gauge fields; and the last two terms are *analogous* (but *not* identical) to torsion-squared terms and products of torsion terms. If one has quaternionic gravity in 8D, the indices are $M, N, L = 1, 2, 3, \ldots, 8$ and, if one wishes, one may build a Lagrangian out of the following tensorial quantities found *within* the quaternionic-valued curvature: namely the 8D Riemannian scalar curvature $\mathcal{R} = g^{(MN)}R_{MN}$, the $U(1)$ and $SU(2)$ field strengths F_{MN}, F^i_{MN}. In particular, let us start with a standard Lagrangian for gravity plus $SU(2) \times U(1)$ Yang–Mills in 8D given by

$$\mathcal{L} = \mathcal{R} - \frac{1}{4}(F_{MN})^2 - \frac{1}{4}(F^i_{MN})^2 \qquad M, N = 1, 2, 3, \ldots, 8 \tag{3.10}$$

where we set the numerical couplings to unity. The components of the Ricci tensors after a Kaluza–Klein compactification are given by [49]

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}K^a_I K_{aJ}F^I_{\mu\rho}F^{J\rho}_{\ \ \nu} \qquad \mathcal{R}_{\mu a} = \frac{1}{2}K^I_a D_\nu F^{I\nu}_{\ \ \mu} \tag{3.11a}$$

$$\mathcal{R}_{ab} = R_{ab} + \frac{1}{4}K^I_a K^J_b F^I_{\mu\nu}F^{J\mu\nu} \tag{3.11b}$$

where K^{aI} are the Killing vectors associated with the $SU(3)$ isometry group (metric preserving symmetry) of the internal space $CP^2 = SU(3)/SU(2) \times U(1)$. The range of the indices is $\mu, \nu = 1, 2, 3, 4$; $a, b = 1, 2, 3, 4$; and $I, J = 1, 2, 3, \ldots, 8$. Equations (2.11*a*) and (2.11*b*) lead to the following decomposition of the 8D scalar curvature

$$\mathcal{R} = R[g_{\mu\nu}] - \frac{1}{4}F^I_{\mu\nu}F^{\mu\nu}_I + g^{ab}R_{ab} + \ldots \tag{3.11c}$$

so that the Lagrangian (3.10) furnishes a 4D theory of gravity and $SU(3)$ Yang–Mills interacting with a nonlinear sigma-model scalar field stemming from the metric degrees of freedom in the internal space. The indices $I = 1, 2, 3, \ldots, 8$ span the eight generators of the $SU(3)$ algebra and $R = g^{(\mu\nu)}R_{\mu\nu}$ is the 4D scalar curvature.

Concluding, from a quaternionic-valued gravitational theory in 8D, one has the necessary field ingredients to build the Lagrangian in (3.10) and generate a gravitational and $SU(3) \times SU(2) \times U(1)$ Yang–Mills theory in 4D after a Kaluza–Klein compactification on CP^2. For this reason, this kind of grand unification program warrants further investigation.

4. A realization of E_8 in terms of $Cl(16) = Cl(8) \otimes Cl(8)$ generators

Note: For convenience, in what follows we are going to use $SO(N)$, $SU(N)$, E_8, … for both the algebras and groups. Mathematicians use $so(N)$, $su(N)$, e_8, … and direct sums, $\oplus$, for Lie algebras; while capital letters and direct products, $\times$, are used for groups. We hope this will not cause confusion. The Lie algebra E_8 is a complex one that admits many different real forms that are described by the difference in the number of non-compact and compact generators. The realization of the E_8 algebra in this section is the one associated with $E_{8(8)}$ with 128 noncompact generators, and 120 compact ones.

The commutation relations of E_8 can be expressed in terms of the 120 $SO(16)$ bivector generators $X^{[IJ]}$ and the 128 $SO(16)$ chiral spinorial generators Y^α as (see ref. 50 and references therein)

$$\begin{aligned}&[X^{IJ}, X^{KL}] = 4(\delta^{IK}X^{LJ} - \delta^{IL}X^{KJ} + \delta^{JK}X^{IL} - \delta^{JL}X^{IK})\\ &\left[X^{IJ}, Y_\alpha\right] = -\frac{1}{2}\Gamma^{[IJ]}_{\alpha\beta}Y^\beta \qquad [Y_\alpha, Y_\beta] = \frac{1}{4}\Gamma^{[IJ]}_{\alpha\beta}X_{IJ}\end{aligned} \tag{4.1a}$$

where $X^{IJ} = -X^{JI}$. It is required to choose a representation of the gamma matrices such that $\Gamma^{[IJ]}_{\alpha\beta} = -\Gamma^{[IJ]}_{\beta\alpha}$ because $[Y_\alpha, Y_\beta]$ is antisymmetric under $\alpha \leftrightarrow \beta$. The Jacobi identities among the triplet $[Y_\alpha, [Y_\beta, Y_\gamma]]$ + cyclic permutation are

$$\Gamma^{IJ}_{\alpha\beta}\Gamma^{IJ}_{\gamma\delta}Y^\delta + \text{ cyclic permutation among } (\alpha, \beta, \gamma) = 0 \tag{4.2a}$$

this Jacobi identity can be shown to be satisfied by contracting two of the spinorial indices (α, β) in (4.2*a*) after multiplying (2.2*a*) by $\Gamma^{\alpha\beta}_{KL}$ and $\Gamma^{\alpha\beta}_{K_1K_2\ldots K_6}$, respectively, giving

$$\Gamma^{IJ}_{\alpha\beta}\Gamma^{\alpha\beta}_{KL}\Gamma^{IJ}_{\gamma\delta} + \Gamma^{IJ}_{\beta\gamma}\Gamma^{\alpha\beta}_{KL}\Gamma^{IJ}_{\alpha\delta} + \Gamma^{IJ}_{\gamma\alpha}\Gamma^{\alpha\beta}_{KL}\Gamma^{IJ}_{\beta\delta} = 0 \tag{4.2b}$$

and

$$\Gamma^{IJ}_{\alpha\beta}\Gamma^{\alpha\beta}_{K_1K_2\ldots K_6}\Gamma^{IJ}_{\gamma\delta} + \Gamma^{IJ}_{\beta\gamma}\Gamma^{\alpha\beta}_{K_1K_2\ldots K_6}\Gamma^{IJ}_{\alpha\delta} + \Gamma^{IJ}_{\gamma\alpha}\Gamma^{\alpha\beta}_{K_1K_2\ldots K_6}\Gamma^{IJ}_{\beta\delta} = 0 \tag{4.2c}$$

Equations (4.2*b*) and (4.2*c*) are zero (which implies that (4.2*a*) is also zero) due to the very special properties of the *chiral* representation of the Clifford gamma matrices in 16D and after decomposing the $(1/2)(128 \times 127) = 8128$ dimensional space of antisymmetric $\Sigma_{[\alpha\beta]}$ matrices into a space involving 120 antisymmetric $\Gamma^{IJ}_{\gamma\delta}$ and 8008 $\Gamma^{I_1I_2\ldots I_6}_{\gamma\delta}$ matrices in their chiral spinorial indices $\gamma\delta$.

The E_8 algebra as a subalgebra of $Cl(16) = Cl(8) \otimes Cl(8)$ is consistent with the $SL(8, R)$ seven-grading decomposition of $E_{8(8)}$ (with 128 noncompact and 120 compact generators) as shown by Koepsell et al. [50]. Such $SL(8, R)$ seven-grading is based on the diagonal part $[SO(8) \times SO(8)]_{\text{diag}} \subset SO(16)$ described in full detail by Koepsell et al. [50].

Baez, in a rigorous detail of the algebra of octonions, described how the 248 generators of E_8 have a 28 + 28 + 3 × (8 × 8) = 248 decomposition consistent with the dimensions of

$$SO\left(V_8^{(1)}\right) \oplus SO\left(V_8^{(2)}\right) \oplus \left(V_8^{(1)} \otimes V_8^{(2)}\right) \oplus \left(S_8^+ \otimes S_8^+\right) \oplus \left(S_8^- \otimes S_8^-\right) \tag{4.3a}$$

where $SO(V_8^{(1)})$ and $SO(V_8^{(2)})$ are two 28-dimensional orthogonal rotation algebras associated with two 8D vector spaces, $V_8^{(1)}$ and $V_8^{(2)}$, respectively. The 16D ($2^{8/2} = 16$) spinor space of $Cl(8)$ is represented by S_8 and decomposes into two invariant subspaces, S_8^+ and S_8^-, forming, the left-handed and right-handed spinor representations of $SO(8)$, respectively [51], and which exhibits triality, as we shall see next, meaning that there is a Z_3 automorphism symmetry that exchanges the 8D vectorial representation V_8 with the S_8^+ and S_8^- left–right 8D spinorial representations.

Pavsic [51] has given a nice interpretation of the two 8D vector spaces $V_8^{(1)}$ and $V_8^{(2)}$ in (4.3*a*) based on the 8, 8 split signature nature of the 16D C-space associated with the 16D (2^4) $Cl(1, 3)$ algebra in 4D Minkowski space–time, and which is comprised of polyvectors of grades 0, 1, 2, 3, 4. The dimension of $SO(16)$ can be decomposed [35] as

$$SO\left(V_8^{(1)}\right) \oplus SO\left(V_8^{(2)}\right) \oplus V_8^{(1)} \otimes V_8^{(2)} \tag{4.3b}$$

spanning the 120 generators X^{IJ}. The tensor products of the spinorial representations $(S_8^+ \otimes S_8^+) \oplus (S_8^- \otimes S_8^-)$ furnish the left-handed 128_+ spinorial representation of $SO(16)$. The other combination $(S_8^+ \otimes S_8^-) \oplus (S_8^- \otimes S_8^+)$ furnishes the right-handed 128_- spinorial representation of $SO(16)$.

A very important remark is in order. Extreme caution must be taken *not* to confuse the seven-grading decomposition of E_8 provided by Larsson, and the actual construction of the 248 generators of E_8, which is provided in this section. Taking the combination of the following tensor products:

$$\left[\gamma^a_{(1)} \oplus \gamma^{a_1a_2}_{(1)} \oplus \gamma^{a_1a_2a_3}_{(1)}\right] \otimes \mathbf{1}_{(2)} + \mathbf{1}_{(1)} \otimes \left[\gamma^b_{(2)} \oplus \gamma^{b_1b_2}_{(2)} \oplus \gamma^{b_1b_2b_3}_{(2)}\right] + \gamma^a_{(1)} \otimes \gamma^b_{(2)} \tag{4.4a}$$

from some of the generators of the two factor $Cl(8)$ algebras, described by the subscripts (1), (2), furnishes Larsson's seven grading of E_8

$$8 + 28 + 56 + 64 + 56 + 28 + 8 = 248 \tag{4.4b}$$

where 8 corresponds to the 8D vectors γ^a, ...; 28 is the 8D bivectors $\gamma^{a_1a_2}$, ...; 56 is the 8D trivector $\gamma^{a_1a_2a_3}$, ...; and $64 = 8 \times 8$ corresponds to the tensor product $\gamma^a_{(1)} \otimes \gamma^b_{(2)}$. However, this *does not* mean that the 248 generators in (4.4*a*) are the actual 248 generators of E_8!. The E_8 generators and its commutators are explicitly constructed next. The set of generators provided by (4.4*a*) does *not* generate an algebra. Their commutators do *not* even close. For example, taking the commutators

$$\left[\gamma_{a_1a_2a_3}, \gamma^{a_4a_5a_6}\right] = 2\gamma^{a_4a_5a_6}_{a_1a_2a_3} - 36\delta^{[a_4a_5}_{[a_1a_2}\gamma^{a_6]}_{a_3]} \tag{4.4c}$$

yields the *sixth*-grade polyvector generator $\gamma^{a_4a_5a_6}_{a_1a_2a_3}$ in the commutators (4.4*c*) and which was *not* initially part of the generators in (4.4*a*). Hence, one can deduce immediately that the latter generators in (4.4*a*) do not constitute a subalgebra. They all are part of the larger algebra $Cl(16) = Cl(8) \otimes Cl(8)$ comprised of polyvectors of grades 0, 1, 2, ..., 16.

We will show later how one can rewrite the E_8 algebra in terms of 8 + 8 vectors Z^a, Z_a ($a = 1, 2, ..., 8$); 28 + 28 bivectors $Z^{[ab]}$, $Z_{[ab]}$; 56 + 56 trivectors $E^{[abc]}$, $E_{[abc]}$; and the $SL(8, R)$ generators, E^b_a, which are expressed in terms of a $8 \times 8 = 64$-component tensor Y^{ab} that can be decomposed into a symmetric part $Y^{(ab)}$ with 36 independent components, and an antisymmetric part $Y^{[ab]}$ with 28 independent components. Its trace $Y^{cc} = N$ yields an element N of the Cartan subalgebra such that the degrees −3, −2, −1, 0, 3, 2, 1 of the seven-grading of $E_{8(8)}$ can be read from ref. 50. We should note that the description of the E_8 generators, later, differs from the one used by Smith [22, 23].

We begin by following ref. 50 very closely and write the full $E_{8(8)}$ commutators in the $SL(8, R)$ basis of ref. 52, after decomposing the $SO(16)$ representations into representations of the subgroup $SO(8) \equiv [SO(8) \times SO(8)]_{\text{diag}} \subset SO(16)$. The indices corresponding to the $\mathbf{8}_v$, $\mathbf{8_s}$, and $\mathbf{8_c}$ representations of $SO(8)$, respectively, will be denoted by a, α, and $\dot\alpha$. After a triality rotation the $SO(8)$ vector and spinor representations decompose as [50]

$$\mathbf{16} \to \mathbf{8_s} \oplus \mathbf{8_c} \tag{4.5}$$

$$\mathbf{128_s} \to (\mathbf{8_s} \otimes \mathbf{8_c}) \oplus (\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{8_v} \oplus \mathbf{56_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \tag{4.6a}$$

$$\mathbf{128_c} \to (\mathbf{8_v} \otimes \mathbf{8_s}) \oplus (\mathbf{8_c} \otimes \mathbf{8_v}) = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{8_c} \oplus \mathbf{56_c} \tag{4.6b}$$

respectively. We thus have $I = (\alpha, \dot\alpha)$ and $A = (\alpha\dot\beta, ab)$, and the E_8 generators decompose as

$$X^{[IJ]} \to (X^{[\alpha\beta]}, X^{[\dot\alpha\dot\beta]}, X^{\alpha\dot\beta}) \qquad Y^A \to (Y^{\alpha\dot\alpha}, Y^{ab}) \tag{4.7}$$

Next we regroup these generators as follows. The 63 generators

$$E^b_a = \frac{1}{8}\left(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} + \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}\right) + Y^{(ab)} - \frac{1}{8}\delta^{ab}Y^{cc} \tag{4.8}$$

for $1 \le a, b \le 8$ span an $SL(8, R)$ subalgebra of E_8. The generator given by the trace $N = Y^{cc}$ extends this subalgebra to $GL(8, R)$. Γ^{ab}, Γ^{abc}, ... are signed sums of antisymmetrized products of gammas. The remainder of the E_8 Lie algebra then decomposes into the following representations of $SL(8, R)$:

$$Z^a = \frac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha}) \tag{4.9a}$$

$$Z_{[ab]} = Z_{ab} = \frac{1}{8}\left(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}\right) + Y^{[ab]} \tag{4.9b}$$

$$E^{[abc]} = E^{abc} = -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha}) \tag{4.9c}$$

and

$$Z_a = -\frac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha}) \tag{4.10a}$$

$$Z^{[ab]} = Z^{ab} = -\frac{1}{8}\left(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}\right) + Y^{[ab]} \tag{4.10b}$$

$$E_{[abc]} = E_{abc} = -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha}) \tag{4.10c}$$

It is important to emphasize that $Z_a \neq \eta_{ab}Z^b$, $Z_{ab} \neq \eta_{ac}\eta_{db}Z^{cd}$, ... and for these reasons one could use the more convenient notation for the generators

$$\mathcal{Z}^a_\pm \equiv \left(Z^a, Z_a\right) \qquad \mathcal{Z}^{ab}_\pm \equiv \left(Z^{ab}, Z_{ab}\right) \qquad \mathcal{Z}^{abc}_\pm \equiv \left(E^{abc}, E_{abc}\right) \tag{4.11}$$

which permits viewing these *doublets* of generators (4.11) as pairs of "canonically conjugate variables", and which in turn, allows us to view their commutation relations as defining a generalized deformed Weyl–Heisenberg algebra with noncommuting coordinates and momenta as shown next. One may define the pairs of complex generators, if one wishes, as

$$V^a = \frac{1}{\sqrt{2}}\left(\mathcal{Z}^a_+ - i\mathcal{Z}^a_-\right) \qquad \overline{V}^a = \frac{1}{\sqrt{2}}\left(\mathcal{Z}^a_+ + i\mathcal{Z}^a_-\right) \tag{4.12a}$$

$$V^{ab} = \frac{1}{\sqrt{2}}\left(\mathcal{Z}^{ab}_+ - i\mathcal{Z}^{ab}_-\right) \qquad \overline{V}^{ab} = \frac{1}{\sqrt{2}}\left(\mathcal{Z}^{ab}_+ + i\mathcal{Z}^{ab}_-\right) \tag{4.12b}$$

$$V^{abc} = \frac{1}{\sqrt{2}}\left(\mathcal{Z}_+^{ab} - i\mathcal{Z}_-^{ab}\right) \qquad \bar{V}^{abc} = \frac{1}{\sqrt{2}}\left(\mathcal{Z}_+^{abc} + i\mathcal{Z}_-^{abc}\right) \tag{4.12c}$$

The remaining $GL(8, R) = Sl(8, R) \times U(1)$ generators are

$$\mathcal{E}^{ab} = \mathcal{E}^{(ab)} + \mathcal{E}^{[ab]} \tag{4.13}$$

The Cartan subalgebra is spanned by the diagonal elements $E_1^1, \ldots, E_7^7$ and N, or, equivalently, by $Y^{11}, \ldots, Y^{88}$. The elements E_a^b for $a < b$ (or $a > b$) together with the elements for $a < b < c$ generate the Borel subalgebra of E_8 associated with the positive (negative) roots of E_8. Furthermore, these generators are graded with respect to the number of times the root α_8 (corresponding to the element N in the Cartan subalgebra) appears, such that for any basis generator X we have $[N, X] = \deg(X) \cdot X$.

The degree can be read off from

$$\begin{aligned} &[N,\ Z^a] = 3Z^a \qquad [N,\ Z_a] = -3Z_a \qquad [N,\ Z_{ab}] = 2Z_{ab} \\ &[N,\ Z^{ab}] = -2Z^{ab} \qquad [N,\ E^{abc}] = E^{abc} \qquad [N,\ E_{abc}] = -E_{abc} \\ &\left[N,\ E_a^b\right] = 0 \end{aligned} \tag{4.14}$$

The remaining commutation relations defining the generalized deformed Weyl–Heisenberg algebra involving pairs of canonical conjugate generators are

$$[Z^a, Z^b] = 0 \qquad [Z_a, Z_b] = 0 \qquad \left[Z_a, Z^b\right] = E_a^b - \frac{3}{8}\delta_a^b N \tag{4.15}$$

This last commutator between the pairs of conjugate Z_a, Z^b generators (like phase space coordinates) yields the deformed Weyl–Heisenberg algebra. The latter algebra is *deformed* because of the presence of the E_a^b generator in the right-hand side of (2.15) and also because the $\mathcal{N}$ trace generator does *not* commute with Z_a, Z^a as seen in (2.14). Similarly, one has the deformed Weyl–Heisenberg algebra among the pairs of conjugate Z_{ab}, Z^{ab} antisymmetric rank-two tensorial generators (like tensorial phase space coordinates in quantum mechanics)

$$[Z_{ab}, Z_{cd}] = 0 \qquad [Z^{ab}, Z^{cd}] = 0 \qquad \left[Z_{ab}, Z^{cd}\right] = 4\delta_{[a}^{[c}E_{b]}^{d]} + \frac{1}{2}\delta_{ab}^{cd}N \tag{4.16}$$

The commutators among the pairs of conjugate and *noncommuting* E_{abc}, E^{abc} antisymmetric rank-three generators (like noncommuting tensorial phase space coordinates) are

$$\begin{aligned} &[E^{abc}, E^{def}] = -\frac{1}{32}\epsilon^{abcdefgh}Z_{gh} \neq 0 \\ &[E_{abc}, E_{def}] = \frac{1}{32}\epsilon_{abcdefgh}Z^{gh} \neq 0 \end{aligned} \tag{4.17}$$

$$\left[E^{abc}, E_{def}\right] = -\frac{1}{8}\delta_{[de}^{[ab}E_{f]}^{c]} - \frac{3}{4}\delta_{def}^{abc}N \tag{4.18}$$

The other commutators among the generalized antisymmetric tensorial generators are

$$\begin{aligned} &\left[Z_{ab}, Z^c\right] = 0 \qquad [Z_{ab}, Z_c] = -E_{abc} \qquad [Z^{ab}, Z^c] = -E^{abc} \\ &\left[Z^{ab}, Z_c\right] = 0 \end{aligned} \tag{4.19}$$

$$\begin{aligned} &[E^{abc}, Z^d] = 0 \qquad \left[E_{abc}, Z^d\right] = 3\delta_{[a}^d Z_{bc]} \\ &\left[E^{abc}, Z_{de}\right] = -6\delta_{de}^{[ab}Z^{c]} \qquad [E_{abc}, Z_{de}] = 0 \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\left[E^{abc}, Z_d\right] = 3\delta_d^{[a}Z^{bc]} \qquad [E_{abc}, Z_d] = 0 \qquad [E^{abc}, Z^{de}] = 0 \\ &\left[E_{abc}, Z^{de}\right] = 6\delta_{[ab}^{de}Z_{c]} \end{aligned} \tag{4.21}$$

The homogeneous commutators among the $GL(8, R)$ generators and those belonging to the deformed Weyl–Heisenberg algebra are

$$\left[E_a^b, Z^c\right] = -\delta_a^c Z^b + \frac{1}{8}\delta_a^b Z^c \qquad \left[E_a^b, Z_c\right] = \delta_c^b Z_a - \frac{1}{8}\delta_a^b Z_c \tag{4.22}$$

$$\begin{aligned} &\left[E_a^b, Z_{cd}\right] = -2\delta_{[c}^b Z_{d]a} - \frac{1}{4}\delta_a^b Z_{cd} \qquad \left[E_a^b, Z^{cd}\right] = 2\delta_a^{[c}Z^{d]b} + \frac{1}{4}\delta_a^b Z^{cd} \\ &\left[E_a^b, E^{cde}\right] = -3\delta_a^{[c}E^{de]b} + \frac{3}{8}\delta_a^b E^{cde} \qquad \left[E_a^b, E_{cde}\right] = 3\delta_{[c}^b E_{de]a} - \frac{3}{8}\delta_a^b E_{cde} \end{aligned} \tag{4.23}$$

Finally, the commutators among the $GL(8, R)$ generators are

$$\left[E_a^b, E_c^d\right] = \delta_c^b E_a^d - \delta_a^d E_c^b \tag{4.24}$$

The elements $\{Z^a, Z_{ab}\}$ (or equivalently $\{Z_a, Z^{ab}\}$) span the maximal 36-dimensional abelian nilpotent subalgebra of E_8 [50], [52]. Finally, the generators are normalized according to the values of the traces given by

$$\begin{aligned} &\mathrm{Tr}(NN) = 60 \cdot 8 \qquad \mathrm{Tr}(Z^a Z_b) = 60\delta_b^a \qquad \mathrm{Tr}(Z^{ab}Z_{cd}) = 60 \cdot 2!\delta_{cd}^{ab} \\ &\mathrm{Tr}\left(E_{abc}E^{def}\right) = 60 \cdot 3!\delta_{abc}^{def} \qquad \mathrm{Tr}\left(E_a^b E_c^d\right) = 60\delta_a^d\delta_c^b - \frac{15}{2}\delta_a^b\delta_c^d \end{aligned} \tag{4.25}$$

with all other traces vanishing.

Using the redefinitions of the generators in (4.11) and (4.12) allows writing the E_8 Hermitian gauge connection associated with the E_8 generators as

$$\begin{aligned} \mathcal{A}_\mu = E_\mu^a V_a + \bar{E}^a \bar{V}_a + E_\mu^{ab}V_{ab} + \bar{E}_\mu^{ab}\bar{V}_{ab} + E_\mu^{abc}V_{abc} + \bar{E}_\mu^{abc}\bar{V}_{abc} \\ + i\Omega_\mu^{(ab)}\mathcal{E}_{(ab)} + \Omega_\mu^{[ab]}\mathcal{E}_{[ab]} \end{aligned} \tag{4.26}$$

where one may set the length scale $L = 1$, scale that is attached to the vielbeins to match the $(length)^{-1}$ dimensions of the connection in (4.26). The $GL(8, R)$ components of the E_8 (Hermitian) gauge connection are the (real-valued symmetric) $\Omega_\mu^{(ab)}$ shear and (real-valued antisymmetric) $\Omega_\mu^{[ab]}$ rotational parts of the $GL(8, R)$ anti-Hermitian gauge connection $i(\Omega_\mu^{(ab)} - i\Omega_\mu^{[ab]})$ such that the $GL(8, R)$ Lie-algebra-valued connection $i\Omega_\mu^{ab}\mathcal{E}_{ab}$ is Hermitian because the $GL(8, R)$ generators $\mathcal{E}_{(ab)}$, $\mathcal{E}_{[ab]}$, and the remaining ones appearing in the E_8 commutators of (4.14)–(4.24), are all chosen to be anti-Hermitian (there are no i factors in the right-hand side of the latter commutators). The (generalized) vielbeins fields are E_μ^a, E_μ^{ab}, E_μ^{abc} plus their complex conjugates. These (generalized) vielbeins fields involving antisymmetric tensorial tangent space indices also appear in generalized gravity in Clifford spaces (C-spaces) where one has polyvector-valued coordinates in the base space and in the tangent space such that the generalized vielbeins are represented by square *and* rectangular matrices [27]. The trace part $\mathcal{N}$ is included in the symmetric shear-like generator $\mathcal{E}_{(ab)}$ of $Gl(8, R)$. The rotational part corresponds to $\mathcal{E}_{[ab]}$.

The E_8 (Hermitian) field strength (in natural units $\hbar = c = 1$) is

$$\mathbf{F}_{\mu\nu} = i[D_\mu, D_\nu] = \left(\partial_\mu\mathcal{A}_\nu^A - \partial_\nu\mathcal{A}_\mu^A + if_{BC}^A\mathcal{A}_\mu^B\mathcal{A}_\nu^C\right)L_A \tag{4.27a}$$

where the indices $A = 1, 2, 3, \ldots, 248$ are spanned by the 248 generators L_A of E_8

$$V_a \quad \bar{V}_a \quad V_{ab} \quad \bar{V}_{ab} \quad V_{abc} \quad \bar{V}_{abc} \quad \mathcal{E}_{(ab)} \quad \mathcal{E}_{[ab]} \tag{4.27b}$$

respectively, giving a total of 8 + 8 + 28 + 28 + 56 + 56 + 36 + 28 = 248 generators.

5. Fermions, E_8, and $Cl(8) \otimes Cl(8)$

In Sect. 1 [12] it was mentioned how an E_8 Yang–Mills in 8D, after a sequence of symmetry-breaking processes based on the *noncompact* forms of exceptional groups, as follows: $E_{8(-24)} \to E_{7(-5)} \times SU(2) \to E_{6(-14)} \times SU(3) \to SO(8, 2) \times U(1)$, leads to a conformal gravitational theory in 8D based on gauging the noncompact conformal group $SO(8, 2)$ in 8D. Upon performing a Kaluza–Klein–Batakis [13] compactification on CP^2, involving a nontrivial *torsion*, which bypasses the no-go theorems that one cannot obtain $SU(3) \times SU(2) \times U(1)$ from a Kaluza–Klein mechanism in 8D, leads to a conformal gravity – Yang–Mills unified theory based on the Standard Model group $SU(3) \times SU(2) \times U(1)$ in 4D. In Sect. 3 it was reviewed how gravity and $SU(3) \times SU(2) \times U(1)$ Yang–Mills in 4D can be obtained from 8D quaternionic gravity after a Kaluza–Klein compactification along the internal CP^2 four-dimensional space [17].

Section 4 was devoted entirely to the algebraic structure of the E_8 algebra and whose 248 Lie algebra generators can be expressed in terms of the generators of the $Cl(16) = Cl(8) \otimes Cl(8)$ Clifford algebra. For this reason, it is not necessary to repeat all the technical details about the $Cl(8)$ algebra.

Let us begin with the first factor $Cl(8)$ in the product $Cl(8) \otimes Cl(8) = Cl(16)$. The 16 × 16 = 256-dim $Cl(0, 8)$ algebra happens to be isomorphic to the $Cl(1, 7)$ algebra, which in turn, is isomorphic to the 16 × 16 real matrix algebra $M(16, R)$. This is relevant in so far that the *noncompact* 8D Lorentz group $SO(1, 7)$ is a subgroup of $Cl(1, 7)$ and whose $[(8 \times 7)/2] = 28$ generators L_{mn} are given by the following Clifford bivectors: $(1/2)\gamma_m \wedge \gamma_n \Rightarrow L_{mn} = (1/4)(\gamma_m, \gamma_n)$, $m, n = 0, 1, 2, \ldots, 7$. The signature corresponding to $Cl(p, q)$ is chosen to be

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^p)^2 - (dx^{p+1})^2 - (dx^{p+2})^2 - \ldots - (dx^{p+q})^2 \tag{5.1}$$

therefore for an 8D space–time one has one temporal coordinate $p = 1$, and seven spatial ones $q = 7$. This fixes the signature to be (+, −, −, …, −). We must remark that the Clifford algebra $Cl(7, 1)$ is isomorphic to the 8 × 8 quaternionic matrix algebra $M(8, \mathbf{H})$ but is *not* isomorphic to the $Cl(1, 7) \sim M(16, R)$ algebras. The algebras $Cl(0, 8) \sim Cl(8, 0) \sim M(16, R)$ are isomorphic. Clifford $Cl(p, q)$ algebras are very sensitive to the signature $p - q$ and dimensions $p + q$ of the space in question. For this reason the existence of Majorana, Weyl, Majorana–Weyl, Symplectic–Weyl spinors, depends on the number of dimensions $p + q$ and the signature $p - q$.

The 28 bivector (anti-Hermitian) generators $L_{mn} = (1/2)\gamma_m \wedge \gamma_n$, of the *second* factor $Cl(8)$ in the product $Cl(8) \otimes Cl(8) = Cl(16)$ obey the $SO(8)$ commutation relations

$$[L_{mn}, L_{pq}] = g_{np}L_{mq} - g_{mp}L_{nq} - g_{nq}L_{mp} + g_{mq}L_{np} \tag{5.2}$$

In Subsect. 2.6 we described how to embed $U(4)$ into $SO(8)$, and such that $U(4) \times U(4) \subset SO(8) \times SO(8)$. The product group $U(4) \times U(4) = SU(4) \times SU(4) \times U(1) \times U(1)$ is large enough to accommodate the Standard Model group $SU(3) \times SU(2) \times U(1)$. As mentioned in Sect. 2, $SU(4)$ is *not* large enough to accommodate the Standard Model group. $SU(4)$ branches as $SU(2) \times SU(2) \times U(1)$ or $SU(3) \times U(1)$ but not into the Standard Model group. Despite that the number of 15 generators of $SU(4)$ is larger than the number of 12 generators (8 + 3 + 1) of the Standard Model group, the rank of $SU(4)$ is three, which is *less* than the rank four of the Standard Model group. $SU(5)$ (rank 4) is large enough to accommodate the Standard Model group. For this reason one needs *two* copies $U(4) \times U(4)$ as explained earlier in Sect. 2 such that one can embed the Standard Model group into $U(4) \times U(4) \subset SO(8) \times SO(8)$.

Another approach to accommodate the Standard Model inside the $Cl(8)$ algebra was already discusssed in Subsect. 2.6. The Standard Model group can be embedded into $SU(5) \subset SO(10)$, and in turn $SO(10)$ can be embedded into *one* copy of the $Cl(8)$ group. The *noncompact* 8D Lorentz group $SO(1, 7)$ is a subgroup of $SO(2, 8)$, which in turn, can be embedded into the $Cl(1, 7) > Cl(0, 8)$ group. Therefore the 8D Lorentz group and conformal group $SO(2, 8)$ can both be embedded into the *second* copy of $Cl(0, 8) \sim Cl(8, 0) \sim M(16, R)$. Consequently, the Standard Model group and the conformal group live inside the *direct* product (not to be confused with the tensor product) $Cl(8) \times Cl(8)$ in such a way that one does not violate the Coleman–Mandula theorem stating roughly that one cannot *mix* space–time symmetries with internal ones. The direct product $Cl(8) \times Cl(8)$ has for dimension $2^8 + 2^8 = 2^9$. Whereas the tensor product $Cl(8) \otimes Cl(8)$ has for dimensions $2^8 \times 2^8 = 2^{16}$ which equals the dimension of $Cl(16)$.

Let us assume that one has three generations of 16 *massless* (chiral) fermions Ψ_α, with each Weyl spinor (half-spinor) having four real components in 4D, the total number of degrees of freedom is then $3 \times 16 \times 4 = 3 \times 8 \times 8 = 192$, which incidentally matches precisely the number $3 \times |\mathbf{O}| \times |\mathbf{O}| = 3 \times 8 \times 8$ where $|\mathbf{O}| = 8$ denotes the real dimension of the octonion algebra. The factor of three is actually due to the *triality* property of $SO(8)$ more than the fact that we have observed three generations. Lisi [8] speculated that because the adjoint and fundamental representation of E_8 are both 248-dimensonal, the massless fermions might correspond to a particular subset of the E_8 gauge fields, and after a symmetry breaking, they acquire masses via the Higgs mechanism. The remaining 56 massless gauge fields will fit into two copies $SO(8) + SO(8)$.

The attempts to recur to this possibility were based in invoking the work of Quillen's superconnection [53]. We shall follow next the arguments of Distler [54] concerning the Quillen superconnection. A typical example of a Quillen superconnection is given by the Lie superalgebra-valued object comprised of a zero-form and one-form

$$D = d + dx^\mu A^a_\mu(x) T_a + \Phi^\alpha(x)\tau_\alpha \qquad d = dx^\mu \frac{\partial}{dx^\mu} \tag{5.3}$$

where T_a, τ_α are the even and odd generators, respectively, of a Lie *superalgebra* and whose (anti) commutators are given by

$$[T_a, T_b] = f^c_{ab}T_c \qquad [T_a, \tau_\alpha] = c^\beta_{a\alpha}\tau_\beta \qquad \{\tau_\alpha, \tau_\beta\} = d^a_{\alpha\beta}T_a \tag{5.4}$$

where $d^a_{\alpha\beta}$ is *symmetric* under the exchange of α, β indices; $A^a_\mu(x)$ and the zero-forms Φ^α (scalar fields) are both bosonic fields; and the curvature of D is defined as $F = [D, D]$ and has an even grade, because the grade of the Quillen's superconnection D by definition is odd (the grade is 1).

Schreiber [54] proposed instead to replace the Lie superalgebra by a Z_2-graded Lie algebra where *all* the generators are *bosonic* and obey the commutators

$$[T_a, T_b] = f^c_{ab}T_c \qquad [T_a, L_\alpha] = c^\beta_{a\alpha}L_\beta \qquad [L_\alpha, L_\beta] = g^a_{\alpha\beta}T_a \tag{5.5}$$

where $g^a_{\alpha\beta}$ is now *antisymmetric* under the exchange of α, β indices. The reason Schreiber wanted to do this is that E_8 is a Lie algebra, and *not* a Lie *superalgebra*, and it admits various Z_2 gradings. The Schreiber superconnection [54] is based on a Z_2-graded Lie algebra given by

$$D = d + dx^\mu A^a_\mu(x) T_a + \psi^\alpha(x) L_\alpha \qquad d = dx^\mu \frac{\partial}{dx^\mu} \tag{5.6}$$

where now ψ^α is a fermionic (anticommuting) Grassmannian-odd field $\psi^\alpha\psi^\beta = -\psi^\beta\psi^\alpha$. Its curvature $F = [D, D]$ differs from the prior case and makes sense because, due to the *antisymmetry* property of $g^a_{\alpha\beta}$ under the exchange of α, β, the curvature contains the product $g^a_{\alpha\beta}\psi^\alpha\psi^\beta \neq 0$, which is not zero.

Schreiber remarked [54] that Lisi [8] was not interested in just any old Z_2-grading of the algebra E_8, but a very particular one. Namely, let us choose some embedding of $SL(2, C)$ into E_8. This defines an action of $SL(2, C)$ on the Lie algebra E_8. One wants the Z_2-grading that comes from the action of the (Z_2) center of $SL(2, C)$ on E_8. Then it is automatic that the Z_2-odd generators transform as spinors of $SL(2, C)$.

Despite this proposal by Schreiber, Distler added that, when all the dust settles, the Schreiber superconnection is equally useless for Lisi's purposes as a Quillen superconnection, though for different reasons as described in full detail by Distler and Garibaldi [55] and which we shall discuss later. Also one should notice that one cannot claim that the space–time chiral fermions Ψ^α can be made to coincide with the anticommuting (Grassmanian-odd) $SL(2, C)$ spinors ψ^α of the Schreiber superconnection (5.6) because the space–time chiral fermion components Ψ^α are commuting $\Psi^\alpha\Psi^\beta = \Psi^\beta\Psi^\alpha$, whereas $\psi^\alpha\psi^\beta = -\psi^\beta\psi^\alpha$ are *anticommuting* and behave differently than the Ψ^α components.

The Quillen superconnection has been used in the construction of internal supersymmetries by Ne'eman and Sternberg [34] to give rise to unified structures that include quarks and leptons. The Quillen superconnection provides a natural setting for the dynamics of an internally supersymmetric theory with the Higgs field occurring as the "zeroth"-order part of the superconnection. The Higgs mechanism enters quadratically into the curvature and hence quarticaly into the Lagrangian. Furthermore, the supercovariant derivatives provide naturally the Yukawa couplings of the Higgs field to the fermions, without having to put them by hand as in the Standard Model [34].

However, the problem here is that there are *no* known Lie superalgebras that are defined similarly to the Lie algebras E_6, E_7, E_8 (e.g., via octonions as in the classical case). The supersymmetric extensions turn out to be *infinite*-dimensional. They belong to the class of affine and hyperbolic Kac–Moody superalgebras like E_9, E_{10}, E_{11} [56]. The infinite dimensional hyperbolic Kac–Moody superalgebras E_{11} have been conjectured by West [31] to encode the hidden symmetries of M-theory in 11 dimensions.

The reason Lie superalgebras could be very appealing to accommodate and incorporate fermions, *geometrically*, is that recently a gauge theory for a (de Sitter – anti-de Sitter) superalgebra that could describe the low-energy particle phenomenology was constructed by Alvarez et al. [57]. The system includes an internal gauge connection one-form $dx^\mu A_\mu$, a spin-1/2 Dirac fermion ψ in the fundamental representation of the internal symmetry group, and a Lorentz connection ω^{ab}. There were many important distinctive features between this theory and standard supersymmetries, in particular that although the supersymmetry is local and gravity is included, there is *no* gravitino and the fermions get their mass from their coupling to the background or from a higher order self-coupling, while bosons remain massless. In four dimensions, following the Townsend–MacDowell–Mansouri construction out of a $osp(4|2)$, $usp(2, 2|1)$ superconnection it produces a Lagrangian invariant under the subalgebra $u(1) \oplus so(3,1)$ and where the only nonstandard additional piece is the Nambu–Jona-Lasinio quartic fermionic terms. In this case, the Lagrangian depends on a single dimensionful parameter that sets the values of Newton's constant, the cosmological constant, and the Nambu–Jona-Lasinio coupling.

Zanelli et al. [57] used the following super-Lie-algebra-vaued connection:

$$\mathbf{A}_\mu = A^A_\mu T_A + \overline{\psi}^\alpha_\mu Q_\alpha + \overline{Q}_\alpha \psi^\alpha_\mu \tag{5.7}$$

where Q_α are the fermionic charge generators and the bosonic ones, T_A, are the $U(1)$ generator, the six Lorentz generators J_{ab}, and four additional generators, J_a, comprising the (anti) de Sitter algebra in four dimensions. Projecting out the gravitino spin-(3/2) component in $\psi^\alpha_\mu \to \psi^\alpha\Gamma_\mu$ leaves only a spin-(1/2) fermion, ψ^α, in (5.7). In this fashion Zanelli at al. [57] recovered a gravitional Lagrangian with a cosmological constant, the Dirac Lagrangian with mass terms plus the couplings of fermions to the background torsion and the Nambu–Jona-Lasinio quartic fermionic terms. We should add also, that there are $(D\overline{\psi})(D\psi)$ terms as well.

The construction of Zanelli et al. [57] could be generalized to supersymmetric extensions of exceptional Lie algebras like E_6, E_7, E_8 but that would involve the use of *infinite*-dimensional affine and hyperbolic Kac–Moody superalgebras like E_9, E_{10}, E_{11}. We leave this project for future work. A unified description of the orthogonal and symplectic Clifford algebras was used recently [58] to construct theories of super-Clifford gravity, super-C-Spaces, higher spins, ..., which might be relevant in generalized supergeometry.

6. Smith's $E_8 \subset Cl(8) \otimes Cl(8)$ algebra-based unification model in 8D

6.1. The Coleman–Mandula theorem and gauge bosons as fermion condensates

We remarked earlier that the Standard Model group and the conformal group in 8D live inside the *direct* product $Cl(8)_{(1)} \times Cl(8)_{(2)}$ in such a way that one does not violate the Coleman–Mandula theorem stating roughly that one cannot mix space–time symmetries with internal ones. Namely that the commutators $[Cl(8)_{(1)}, Cl(8)_{(2)}] = 0$. However when using tensor products $Cl(8)_{(1)} \otimes Cl(8)_{(2)}$ in the construction of E_8, one has to check that the commutators of the tensor products of the matrix representations of the $SO(8)$ $(SO(1, 7))$ bivector generators and the unit element represented by matrix $\mathbf{1}$ also vanish. Such commutators can be written symbolically as $[SO(1, 7) \otimes \mathbf{1}, \mathbf{1} \otimes SO(8)]$ and we must check that they vanish.

After some straightforward algebra one can verify that the fundamental identities

$$[A \otimes B, C \otimes D] = \frac{1}{2}[A, C] \otimes \{B, D\} + \frac{1}{2}\{A, C\} \otimes [B, D] \tag{6.1a}$$

and

$$\{A \otimes B, C \otimes D\} = \frac{1}{2}\{A, C\} \otimes \{B, D\} + \frac{1}{2}[A, C] \otimes [B, D] \tag{6.1b}$$

are a direct *consequence* of the definition

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{6.1c}$$

Therefore from (6.1a) one has

$$[SO(1, 7) \otimes \mathbf{1}, \mathbf{1} \otimes SO(8)] = \frac{1}{2}[SO(1, 7), \mathbf{1}] \otimes \{\mathbf{1}, SO(8)\} + \frac{1}{2}\{SO(1, 7), \mathbf{1}\} \otimes [\mathbf{1}, SO(8)] = 0 \tag{6.2}$$

and there will be *no* mixing between the space–time symmetries and the internal ones because of the vanishing contribution of all the terms in the right-hand side of (6.2). Thus, one does not violate the Coleman–Mandula theorem.

However if one looks at the explicit E_8 commutation relations of Sect. 4 among the bivector generators $[Z_{ab}, Z^{cd}] \neq 0$ we can see these are *not* vanishing. The E_8 lives inside the *tensor* product $Cl(8)_{(1)} \otimes Cl(8)_{(2)}$. However when all of the E_8 generators E^a_b, Z_a, Z^a, Z_{ab}, Z^{ab}, Z_{abc}, Z^{abc}

are given by very specific linear combinations of the 120 bosonic $X^{[IJ]}$, and 128 bosonic Y^A $SO(16)$ generators as displayed in Sect. 4, there will be entanglement among Z_{ab} and Z^{ab} and there will be mixing of space–time (external) symmetry generators and internal symmetry ones. A close inspection of (4.9*b*), (4.10*b*), and (4.16) reveals that this is the case because the two sets of $SO(8)$ bivector generators *do not commute* $[Z_{ab}, Z^{cd}] \neq 0$ so that there is *mixing* among external and internal symmetries.

The question is now how does one resolve the discrepancy with the results in (6.2)? The source of the discrepancy has to do with the choice of basis for the E_8 algebra. A tensor product of $Cl(8)_{(1)} \otimes Cl(8)_{(2)} = Cl(16)$ produces 2^{16} generators. Out of this very large number of generators one must extract a linearly independent basis of 248 E_8 generators that is compatible with the basis description outlined in Sect. 4 and based on the 120 bosonic $X^{[IJ]}$ and 128 bosonic Y^A $SO(16)$ generators.

As we emphasized earlier, the set of 248 generators used in (4.4*a*) to describe the seven-grading of the E_8 algebra by Larsson did *not* constitute an algebra (subalgebra of $Cl(16)$) because their commutators do *not* close as shown in (4.4*c*). This means, in particular, that the bivector $SO(1,7) \otimes \mathbf{1}$ and $\mathbf{1} \otimes SO(8)$ generators, and the other generators of (4.4*a*), *are not* given by suitable linear combinations of the bivector generators Z_{ab}, Z^{ab} and the other E_8 generators in (4.8)–(4.10). The reason being that the latter E_8 generators constitue an algebra, while the former generators in (4.4*a*) do *not*. This is the underlying reason why $[SO(1,7) \otimes \mathbf{1}, \mathbf{1} \otimes SO(8)] = 0$ and $[Z_{ab}, Z^{cd}] \neq 0$.

A change of basis for the E_8 generators is in principle possible such that in the new basis the commutators are $[Z'_{ab}, Z'^{cd}] = 0$ and there would not be an entanglement. For this reason we believe that it would be simpler from the beginning to focus on the *direct* product $Cl(8)_{(1)} \times Cl(8)_{(2)}$ in such a way that one does not violate the Coleman–Mandula theorem, like we did in Sect. 2 when dealing with the direct product of $Cl(4, C) \times Cl(4, C) = Cl(5, C)$ (at the algebra level one has $cl(5, C) = cl(4, C) \oplus cl(4, C)$).

Because $SO(1,4) \times SO(3) \subset SO(1,7)$, the $SO(1,7)$ bivector generators contains: (*i*) the de Sitter group $SO(1,4)$ in a 4D space–time, which is the one associated with the MMCW formulation of gravity as shown in (2.12) and (2.13); and (*ii*) the $SO(3) \sim SU(2)$. Because $SO(6) \times SO(2) \subset SO(8)$, the $SO(8)$ bivector generators contain: (*i*) $SO(6) \sim SU(4)$; and (*ii*) $SO(2) \sim U(1)$. Hence, the combination of the $SO(1,7)$ and $SO(8)$ bivector generators contain the de Sitter group $SO(1,4)$ in 4D space–time, $SU(2)$ and $SU(4) \times U(1)$ (the total combination is large enough to contain the Standard Model via the branching $SU(4) \to SU(3) \times U(1)$).

Other important remarks are in order. Let us look at the 120 + 128 generators of the $SO(16)$ algebra in (4.7), which led to the construction of all the E_8 generators E^a_b, Z_a, Z^a, Z_{ab}, Z^{ab}, Z_{abc}, Z^{abc}. The first 120 = 28 + 28 + 64 generators are, respectively,

$$X^{[IJ]} \to (X^{[\alpha\beta]}, X^{[\dot\alpha\dot\beta]}, X^{\alpha\dot\beta}) \qquad \alpha, \beta = 1, 2, \cdots, 8 \qquad \dot\alpha, \dot\beta = 1, 2, \cdots, 8 \tag{6.3}$$

the remaining 128 = 64 + 64 bosonic generators obey commutation relations despite the index A being $Spin_+(16)$ chiral spinorial, are

$$Y^A \to Y^{\alpha\dot\alpha}, Y^{ab} \qquad a, b = 1, 2, \ldots, 8 \tag{6.4}$$

The 28 + 28 bivectors associated to two copies of $SO(8)$ are given by $X^{[\alpha\beta]}$ and $X^{[\dot\alpha\dot\beta]}$, respectively.

Focusing now on the triplet set of 64 generators each, and given by

$$X^{\alpha\dot\beta}, \; Y^{\alpha\dot\alpha}, \; Y^{ab} \tag{6.5}$$

one could try to reinterpret two sets of 8 × 8 = 64 massless spin-1 bosonic gauge fields (associated with the first two sets of 64 generators described previously) as if they were *massless* "fermion-composites" of two spin-1/2 massless fermions; that is, comprised of massless fermion–antifermion pairs, such as $A_\mu \sim \overline{\Psi}\gamma_\mu\Psi$, omitting internal E_8 indices and space–time spinorial ones.

The massless antifermions have opposite chiralities to the fermions so the $\mathbf{8}_c$ spinorial representation associated with an antifermion corresponds actually to the $\mathbf{8}_s$ spinorial representation of the fermion counterpart, and vice versa. Hence, two of the sets of 64 generators associated with the *massless* fermion–antifermion "condensates" correspond to the tensor products of the $SO(8)$ spinorial representations $\mathbf{8}_c \otimes \mathbf{8}_c$ and $\mathbf{8}_s \otimes \mathbf{8}_s$ as in Lisi's model [8].

The fermions in Smith's model are assembled into octet-multiplets associated with the eight octonion basis elements e_0, e_1, e_2, …, e_7 and which correspond in the first generation to the electron neutrino ν_e; the red, blue, and green up quarks u^r, u^b, u^g; the electron e; and the red, blue, and green down quark d^r, d^b, d^g, respectively. Their respective antiparticles fall into another octonion-multiplet. The problem is that these fermions are *not* massless. Hence one cannot use them as candidates for the fermion-condensates, unless one assumes them to be massless and later gain their mass via the Higgs mechanism. There are other problems as well, even if they are massless as occurred with the neutrino theory of light [59].

There is the third set Y^{ab} of 64 = 8 × 8 generators corresponding to the tensor product of two $SO(8)$ vector representations $\mathbf{8}_v \otimes \mathbf{8}_v$. Smith interprets the 64 spin-1 bosonic gauge fields (associated with this set Y^{ab} of 64 generators) as if they were bilinears $X^\mu P^\nu$ involving the 8D space–time X^μ coordinates plus their momentum P^ν conjugates. Because the classical phase-space coordinates are not operators, because X and P commute, one needs another interpretation. More rigorously, one could say that a realization of the 64D $U(8)$ algebra bilinear in the fermionic oscillators as described in Sect. 2.6, $E^j_i = a^\dagger_i a_j$, $i, j = 1, 2, 3, \ldots, 8$, and obeying the commutators $[E^j_i, E^l_k] = \delta^j_k E^l_i - \delta^l_i E^j_k$ seems closer to our goals.

However, to be physically rigorous, one must emphasize that the triplet set of 64 spin-1 gauge fields (bosons) associated with the generators $X^{\alpha\dot\beta}$, $Y^{\alpha\dot\alpha}$, Y^{ab} are *not* massless fermion-composites, *nor* phase-space coordinates composites, but just mere spin-1 bosons (gauge fields). All of the E_8 gauge fields must be *fundamental*.

Furthermore, let us suppose, for the sake of argument, that one generation of massless fermions allowed us to generate 128 gauge bosons as fermion–antifermion condensates. We still have two more generations of fermions whose fermion–antifermion condensates would make up two sets of additional 128 bosons. This is very problematic because there is *no* room for 256 extra gauge fields inside E_8.

The neutrino theory of light was proposed in 1932 by L. de Broglie who suggested that the photon might be the combination of a neutrino and an antineutrino. Pryce showed that one cannot obtain both Bose–Einstein statistics and transversely polarized photons from neutrino–antineutrino pairs [59]. There is convincing evidence that neutrinos have mass. In experiments at the Super Kamiokande, researchers [59] have discovered neutrino oscillations in which one flavor of neutrino changed into another. This means that neutrinos have nonzero mass. Because massless neutrinos are needed to form a massless photon, a composite photon is not possible.

6.2. Octonionic realization of *GL*(8, *R*) and *SU*(3) color algebra of quarks

The octonionic algebra, being nonassociative, is difficult to manipulate. The authors [18] introduced left–right octonionic barred operators, by acting on the left and right on octonionic-valued functions (comprised of eight entries), and which enabled them to find a realization of the associative $GL(8, R)$ group in terms of 8 × 8 matrices. Octonionic realizations of the four-dimensional Clifford algebra and $GL(4, C)$ were also constructed. Dixon [60] has explicitly displayed the octonionic realizations of $SU(3)$ and G_2 in terms of linear combinations of suitable bilinear products of left-acting operators.

The left-barred operators act on octonionic valued functions Ψ as $[\mathbf{a})\mathbf{b}]\Psi = (\mathbf{a}\Psi)\mathbf{b}$. The right-barred operators act on octonionic valued functions Ψ as $[\mathbf{a}(\mathbf{b}]\Psi = \mathbf{a}(\Psi\mathbf{b})$. One has $\Psi = \Psi_o e_o + \Psi^i e_i$ and $\mathbf{a} = a_o e_o + a^i e_i$, $\mathbf{b} = b_o e_o + b^i e_i$. The octonion basis elements e_o, e_i, $i = 1, 2, 3, \ldots, 7$ obey the relations $e_i e_j = -\delta_{ij} e_o + c_{ijk} e_k$ where the structure constants c_{ijk} are fully antisymmetric in their indices. e_o is the unit element and e_i are the seven octonion imaginary units. For the octonionic imaginary units one has that the associator $\{e_i, e_j, e_k\} = (e_i e_j)e_k - e_i(e_j e_k) = 2d_{ijkl} e_l$ does *not* vanish because of the nonassociative nature of the octonion algebra.

Defining the left-action (corresponding to the seven imaginary elements e_m) by L_m, $m = 1, 2, \ldots, 7$, and the right-action (corresponding to the seven imaginary elements e_n) by R_n, $n = 1, 2, \ldots, 7$ one can find a realization of L_m, R_n in terms of 8×8 matrices and extract two different bases for $GL(8, R)$. One basis is comprised of $\mathbf{1}$, L_m, R_n, $R_n L_m$ giving a total of $1 + 7 + 7 + 49 = 64$ (8×8) matrices representing $GL(8, R)$. Another basis is $\mathbf{1}$, L_m, R_n, $L_m R_n$ giving a total of $1 + 7 + 7 + 49 = 64$ (8×8) matrices. This provides a one-to-one correspondence between the left–right barred octonion operators and $GL(8, R)$. The authors [18] also showed that

$$\begin{aligned} L_m L_n &= -\delta_{mn} + c_{mnp} L_p + [R_n, L_m] \\ R_n R_m &= -\delta_{mn} + c_{mnp} R_p + [L_m, R_n] \end{aligned} \tag{6.6}$$

By introducing a new matrix multiplication defined in terms of ordinary matrix multiplication as

$$L_m * L_n = L_m L_n - [R_n, L_m] \quad \Rightarrow \quad L_m * L_n = -\delta_{mn} + c_{mnp} L_p \tag{6.7}$$

one reproduces the nonassociative and noncommutative octonionic algebra.

An octonionic representation for the Dirac Hamiltonian was given by De Leo and co-workers [18]. The complexified octonionic solutions found by using the complex inner products defined in ref. 18 contain two orthogonal spinorial solutions, Ψ_1, Ψ_2, and each solution with its four complex degrees of freedom represent a Dirac particle. This suggests a natural simple *one*-dimensional octonionic formulation of the Standard Model, where two orthogonal spinorial solutions are needed to represent the leptonic and quark doublets [61].

The split-octonion algebra is based on the choice of basis

$$\begin{aligned} &u_0 = \frac{1}{2}(e_0 + ie_7) \qquad u_0^* = \frac{1}{2}(e_0 - ie_7) \qquad u_i = \frac{1}{2}(e_i + ie_{i+3}) \\ &u_i^* = \frac{1}{2}(e_i - ie_{i+3}) \end{aligned} \tag{6.8}$$

for $i = 1, 2, 3$. One learns that u_i, u_i^*, for $i = 1, 2, 3$, behave like fermionic creation and annihilation oscillators corresponding to an exceptional nonassociative Grassmannian algebra

$$\begin{aligned} &\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0 \qquad \{u_i, u_j^*\} = -\delta_{ij} \qquad i, j = 1, 2, 3 \\ &\frac{1}{2}[u_i, u_j] = \epsilon_{ijk} u_k^* \qquad \frac{1}{2}[u_i^*, u_j^*] = \epsilon_{ijk} u_k \qquad (u_0)^2 = u_0 \\ &(u_0^*)^2 = u_0^* \end{aligned} \tag{6.9}$$

Unlike the octonion algebra, the split-octonion algebra is *not* a division algebra because it contains *zero* divisors.

The automorphism group of the octonion algebra is the 14D G_2. It admits $SU(3)$ as the subgroup leaving invariant the e_7 imaginary element and the idempotents u_0, u_0^*. Gursey and Gunaydin [19] identified this $SU(3)$ as the color group acting on the quark and antiquark triplets $\Psi_\alpha = u_i \Psi^i_\alpha$; $\overline{\Psi}_\alpha = -u_i^* \overline{\Psi}^i_\alpha$, $i = 1, 2, 3$. From the split-octonion multiplication table one learns that triplet × triplet = antitriplet; antitriplet × antitriplet = triplet; and triplet × antitriplet = singlet, providing a very natural algebraic interpretation of quark confinement. Mesons are comprised of a quark–antiquark pair, while (anti) baryons are comprised of three (anti)quarks. This preamble is necessary to understand the use of octonions in what follows.

6.3. The Lagrangian in Smith's physics model

Smith's physical model is based on a 4D Lagrangian that has its origins in a parent 8D theory based on a gauge theory associated with the Clifford group $Cl(8) \otimes Cl(8) = Cl(16)$ (the isomorphism is due to the eight-fold periodicity of real Clifford algebras). The 4D Lagrangian is obtained after a spontaneous compactification process from eight to four dimensions is performed. One must not confuse a Kaluza–Klein spontaneous compactification mechanism with a dimensional reduction. A higher-dimensional universe with compactified extra dimensions admits a four-dimensional description consisting of an *infinite* Kaluza–Klein tower of fields [62]. At lower energies one does not see that infinite tower of fields.

The group $U(4) \subset SO(8)$ is used to get the color group $SU(3)$, while the $U(2) = SU(2) \times U(1)$ emerges from the isotropy group in $SU(3)/U(2)$ defining the coset internal space CP^2 and is based on the Kaluza–Klein–Batakis mechanism (requiring torsion) obtained from an spontaneous compactification of $M_8 \rightarrow M_4 \times CP^2$. The other pseudo-unitary group $U(2, 2) \subset SO(1, 7)$ living in the second copy of $SO(1, 7) \subset Cl(1, 7) > Cl(0, 8)$ is needed to obtain a $SU(2, 2)$ conformal gauge theory of gravity in four dimensions.

The selected terms in the 4D Lagrangian (there are *many* other terms in the E_8 parent gauge field theory in 8D) is comprises the following four pieces:

1. In 4D, when there is self-duality $F = {}^*F$; the Yang–Mills Lagrangian $\mathrm{Tr}(F \wedge^* F)$ becomes $\mathrm{Tr}(F \wedge F)$, which is the basis to build the MMCW Lagrangian associated with the $U(2, 2) = SU(2, 2) \times U(1)$ algebra as described in (2.12) and (2.13). The MMCW action is the one used by Smith [22, 23] to account for gravity.

   We should notice that in 8D the natural object upon which one builds an action is the eight-form $\langle F \wedge F \wedge F \wedge F\rangle$ where the $\langle\,\rangle$ symbol denotes extracting the group invariant element among the wedge product and requires an invariant group-tensor to contract group indices. In $D = 16$, the natural object will be the 16-form made out of 8 factors $\langle F \wedge F \wedge \ldots F \wedge F\rangle$. This is how a Chern–Simons E_8 gauge theory of gravity, based on the octic E_8 invariant construction by Cederwall and Palmkvist [11], was used by Castro [12] to build a unified field theory (at the Planck scale) of a Lanczos–Lovelock gravitational theory with a E_8 generalized Yang–Mills field theory and which is defined in the 15D boundary of a 16D bulk space.
2. A Yang–Mills Lagrangian associated with the $SU(3) \times SU(2) \times U(1)$ group.

$$-\frac{1}{4}\left(Tr_{SU(3)}\left[F_{\mu\nu}F^{\mu\nu}\right] + Tr_{SU(2)}\left[F_{\mu\nu}F^{\mu\nu}\right] + \left[F_{\mu\nu}F^{\mu\nu}\right]_{U(1)}\right) \tag{6.10}$$

3. A Ginzburg–Landau–Higgs term

$$-\left(D_\mu\Phi^\dagger\right)(D^\mu\Phi) - \frac{1}{4}\lambda(\Phi^\dagger\Phi)^2 + \frac{1}{2}m^2\Phi^\dagger\Phi \tag{6.11}$$

The complex scalar field Φ is an $SU(2)_L$ doublet; and $\Phi^\dagger$ is the Hermitian adjoint. The complex scalar field terms originate from the dimensional reduction to 4D of the 8D Yang–Mills action

$$-\frac{1}{4}\int_{M^8} \mathrm{Tr}[\mathbf{F} \wedge^* \mathbf{F}] \tag{6.12}$$

via the Mayer–Trautman mechanism [63]. The Ni–Lou–Lu–Yang method [64] is used to calculate the Higgs mass.

4. In the Standard Model, the Dirac mass terms for the fermions are generated after Yukawa couplings among the leptons and quarks with the Higgs field are introduced, and the mechanism of spontaneous symmetry breaking has been used. In Smith's model, the 8D Lagrangian integral is such that the mass m emerges from the internal space kinetic terms $\overline{\Psi}\gamma^a D_a \Psi = \overline{\Psi}\gamma^a \partial_a \Psi + \overline{\Psi}\gamma^a A_a \Psi$; $a = 1, 2, 3, 4$ and which represent the internal 4D space contribution to the 8D Dirac kinetic terms $\overline{\Psi}\gamma^M D_M \Psi M = 1, 2, \ldots, 8$. The four internal components $A_a = A_1, A_2, A_3, A_4$ of the gauge fields behave like four scalars from the 4D space–time point of view. Those four real scalars can be assembled into two complex scalars that represent the complex Higgs $SU(2)$ doublet Φ. Thus from $\overline{\Psi}\gamma^a A_a \Psi \sim \overline{\Psi}\Phi\Psi$ one will generate Yukawa-type couplings leading to mass terms for the fermions when Φ acquires a VEV.

In Sect. 5 we discussed the work of Alvarez et al. [57], which generates Dirac mass terms geometrically in 4D and directly from the coupling of the fermions to the background geometry. The Standard Model fermionic kinetic terms $\Sigma_f \overline{\Psi}_f \gamma^\mu D_\mu \Psi_f$ and mass terms $\sum_f m_f \overline{\Psi}_f \Psi_f = \Sigma_f m_f (\overline{\Psi}_{f,\mathrm{L}} \Psi_{f,\mathrm{R}} + \overline{\Psi}_{f,\mathrm{R}} \Psi_{f,\mathrm{L}})$ involve a summation over the three generations of chiral fermions, Weyl spinors. (Mathematicians use the terminology of half-spinors, here we shall use the physicist terminology.) Each family (generation) comprises 16 fermions, as described in (2.28), once a *massive* neutrino is introduced comprising both a left- and right-handed component $\Psi_{\mathrm{R,L}} = (1/2)(1 \pm \gamma_5)\Psi$.

The fermion assignment by Smith differs from the one described in (2.28) and (2.29). It is connected to the octonion-multiplet associated with the eight octonion basis elements and which correspond, respectively, to the electron neutrino, ν_e; the red, blue, and green up quarks u^r, u^b, u^g; the electron, e; and the red, blue, and green down quarks d^r, d^b, d^g. The antiparticles fall into another octonion-multiplet. At low energies (where we do experiments) a quaternionic structure freezes out, splitting the 8D space–time into a 4D physical space–time M_4 and a 4D internal symmetry space CP^2.

The first generation of fermion particles is represented by octonions. The first generation of fermion antiparticles is represented by octonions in a similiar way. The second generation of fermion particles and antiparticles are represented by pairs of octonions. The third generation of fermion particles and antiparticles are represented by triples of octonions. Because the octonions are nonassociative one must not confuse a triplet of octonions (X_1, X_2, X_3) with the triple products $X_1(X_2X_3) \neq (X_1X_2)X_3$. This representation of the fermion families is the basis of the combinatorics used in the fermion mass calculations [22, 23] to be discussed in Sect. 7. In the next section we shall focus on the existence of chiral fermions after compactifications to lower dimensions.

6.4. Chiral fermions and instanton backgrounds in CP^n

The complex projective space $CP^2 = SU(3)/U(2)$ was actively investigated in the 1980s as an interesting candidate for an Euclidean gravitational instanton. The Euler characteristic of CP^2 is three ($n + 1$ for CP^n) and the Hirzebruch signature is one. It is *not* a spin manifold; there is a global obstruction to putting spinors on this space, because the second Stiefel–Whitney class is not zero. CP^n admits globally defined spinors for *odd n*, but *not* for *even n*. However, one can still put spinors on it, provided fundamental gauge fields are added; namely, if an appropriate topologically nontrivial background gauge field is introduced. This fact was used in ref. 65 to construct a generalised spin structure, *Spin*^{c}, where spinors with an Abelian charge move in the field of the Kahler two-form on CP^2, which is somewhat analogous to a monopole field on $CP^1 = S^2$. To sum up, we have the interpretation of *Spin*^{c} structures as being (locally) a spinor with an attendant $U(1)$ gauge connection. One may also construct *Spin*^{c} structures associated with nonabelian fields as well, by including topologically nontrivial Yang–Mills gauge fields on CP^n.

It was shown by Dolan and Nash [20] that the quarks and leptons of the Standard Model, including a right-handed neutrino, can be obtained by gauging the holonomy groups of complex projective spaces of complex dimensions two and three. The spectrum emerges as chiral zero modes of the Dirac operator coupled to gauge fields and the demonstration involves an index theorem analysis on a general complex projective space in the presence of topologically nontrivial $SU(n) \times U(1)$ gauge fields.

The electroweak sector of the Standard Model emerges naturally in this construction from $CP^2 = SU(3)/U(2)$ when the gauge group is taken now [20] to be the holonomy group $U(2)$, instead of the $SU(3)$ isometry group, and the usual *Spin*^{c} structure gives rise to a neutral singlet which is identified with the right-handed neutrino, while tensoring the standard *Spin*^{c} bundle with the inverse of the canonical line bundle gives another $SU(2)$ singlet with the quantum numbers of the right-handed electron. The electron–neutrino doublet arises by coupling spinors to a natural rank 2 bundle, which is dual to the generating line bundle. The curvature associated with this bundle represents a $U(2)$ instanton on CP^2.

A very rigorous application of the Atiyah–Singer index theorem for fermions coupled to gauge fields in CP^n backgrounds was used by Dolan and Nash [20] to determine the number of chiral zero (massless) modes of the (generalized) Dirac operator; that is, the number of positive chirality zero modes minus the number of negative chirality zero modes equals the index that determines the number of fermion generations.

For a $SU(n)$ singlet with $U(1)$ charge $Y = q$, where q is an *integer*, the index in CP^n is [20]

$$\nu_q = \frac{1}{n!}(q + 1)(q + 2)\ldots(q + n) \tag{6.13}$$

A fermion in the fundamental **n**D representation of $SU(n)$, with a $U(1)$ charge $Y = q + (1/n)$, has an index given by

$$\nu_{q,n} = \frac{(q + n + 1)(q + 1)(q + 2)\ldots[q + (n - 1)]}{(n - 1)!} \tag{6.14}$$

On CP^2 Dolan and Nash [20] had: (*i*) an $SU(2)$ singlet with $q = 0$ giving zero charge $Y = 0$ and index $\nu_0 = +1$; (*ii*) a second $SU(2)$ singlet with $q = -3$ giving charge $Y = -3$ and index $\nu_{-3} = +1$; and (*iii*) an $SU(2)$ doublet with $q = -2$ giving charge $Y = -2 + (1/2) = (-3/2)$ and index $\nu_{q=-2,n=2} = -1$.

On CP^3 Dolan and Nash [20] had: (*i*) An $SU(3)$ singlet with $q = 0$ giving a charge $Y = 0$ and index $\nu_0 = 1$; (*ii*) An $SU(3)$ triplet with $q = -3$ giving a charge $Y = -3 + (1/3) = -(8/3)$ and index $\nu_{q=-3,n=3} = 1$.

Interpreting positive (negative) index as giving right- (left)-handed spinors, and rescaling the Y charge by 2/3 (Dolan and Nash scaled it by 1/3), this results for CP^2 in a single generation of particles of the electroweak sector of the standard model, including a right-handed neutrino. There are two $SU(2)$ singlets and one $SU(2)$ doublet given, for example, by a right-handed electron neutrino: a right-handed electron, and a left-handed doublet comprising an electron neutrino and an electron as follows:

$$\mathbf{1}_0 = \nu_{\mathrm{e,R}} \qquad \mathbf{1}_{-2} = \mathrm{e_R} \qquad \mathbf{2}_{-1} = (\nu_{\mathrm{e,L}}, \mathrm{e_L}) \tag{6.15}$$

The subscripts denote their weak charges, Y, and the normalization is such that the electric charge agrees now with the conventional form $Q = I_3 + (Y/2)$. For example, for the doublet, we have the third component of isospin $I_3(\nu_{e,L}) = (1/2) \Rightarrow Q = (1/2) - (1/2) = 0$ and $I_3(e_L) = -(1/2) \Rightarrow Q = -(1/2) - (1/2) = -1$. Under CPT conjugation one gets their antiparticles: a left-handed electron antineutrino, $\bar{\nu}_{e,L}$; a left-handed positron, $\bar{e}_L$; and a right-handed doublet comprising an electron antineutrino and a positron, $(\bar{\nu}_{e,R}, \bar{e}_R)$.

The results for CP^3 revealed [20] (after scaling the Y charges by suitable factors) one complete generation of the quark sector of the standard model. For example, the right-handed up quark u_R; the right-handed down quark d_R, and a left-handed doublet (u_L, d_L). Under CPT conjugation one gets their antiparticles: the left-handed up antiquark $\bar{u}_L$; the left-handed down antiquark $\bar{d}_L$, and a right-handed doublet of antiquarks $(\bar{u}_R, \bar{d}_R)$.

A single complete generation of the Standard Model was obtained successfully by Dolan and Nash [20]. The generalized *Spin^c* structures were described in terms of tensor products of the exterior bundle of antiholomorphic k forms in CP^2, CP^3 with powers of $U(1)$ line bundles and higher rank n vector bundles. (See ref. 20 for full details.)

However, a number of questions and problems are still present. Firstly there is *no* obvious sign of *three* generations. Inserting different positive and negative integer values of q into index formulas (6.13) and (6.14) would yield different values for the number of generations, but the fermions no longer carry the correct quantum numbers of the Standard Model. Dolan and Nash argued that one could obtain more generations by taking copies of CP^2, but there seems no compelling reason to take three such copies and not some other number. Secondly, because the internal manifold $CP^2 \times CP^3$ is 10-dimensional, and space–time is 4D, the total space–time has 14 dimensions, which is riddled with quantum anomalies.

They also remarked that this issue may be related to the question of what possible role the isometry group may play. In particular, they added that the smallest nontrivial matrix approximation to CP^2 is the algebra of 3×3 matrices, acting on a three-dimensional complex vector space, which carries the fundamental representation of the isometry group $SU(3)$, and it may be that this could be interpreted as a horizontal symmetry giving rise to three generations [20].

Note that the philosophy here is rather different than the usual Kaluza–Klein approach where the isometry group is identified with the gauge group. In the work of Dolan and Nash [20], the isometry group is identified with a horizontal symmetry group and the holonomy group is the gauge group. On CP^2 one has $SU(3)$ and, using this as a horizontal generation group, the fundamental representation would give three generations. But then it is not clear what the role of the $SU(4)$ from $CP^3 = SU(4)/U(3)$ would be.

Distler and Garibaldi published a critical paper [55] arguing that Lisi's E_8 "theory of everything" [8] in four dimensions, and a large class of related models, cannot work. They offered a direct proof that it is impossible to embed all three generations of fermions in E_8, or to obtain even the one-generation Standard Model without the presence of an antigeneration comprising *mirror* fermions (fermions carrying opposite chirality to ordinary fermions). Other problems were cited by Motl [66] objecting to the addition of bosons and fermions in Lisi's superconnection, and to the violation of the Coleman–Mandula theorem. Lisi, Smolin, and Simone Speziale [67] later on proposed an action- and symmetry-breaking mechanism, and used an alternative treatment of fermions.

Chakraborty and Parthasarathy [68], following the work of Hawking and Pope [65], have shown how an $U(1)$ instanton field configuration on CP^2 triggered a compactification from eight to four dimensions, $M_8 \to M_4 \times CP^2$, and it led to an integer-valued index, but to half-integer values for the electric charges of the chiral fermions. Their action was based on a $U(1)$ Maxwell gauge field, plus gravity and a cosmological constant. Despite half-integer charges appearing in the results of ref. 68 for the $U(1)$ instanton, integer-valued charges may occur for non-Abelian gauge fields coupled to fermions due to a nontrivial topological twist generating an extra 1/2 contribution to the electric charge.

It is required to repeat the Chakraborty and Parthasarathy's [68] construction and index calculation for the $SU(4) \subset SO(8)$ gauge field and verify (via a rigorous mathematical calculation of the Atiyah–Singer index) whether or not it leads to three generations with the right $SU(3) \times SU(2) \times U(1)$ quantum numbers for all the leptons and quarks, in particular, to check that the electric charge is integer-valued. According to Smith [23] it is $SO(8)$ that acts on the CP^2 internal part of $M_4 \times CP^2$ through its $SU(4)$ subalgebra that contains the color $SU(3)$, while the electroweak $U(2) = SU(2) \times U(1)$ originates from the isotropy $U(2)$ group in $CP^2 = SU(3)/U(2)$ via the Batakis mechanism [13].

Without the actual calculation of the Atiyah–Singer index as it was rigorously performed by Dolan and Nash [20] on CP^2, for example, one cannot claim with absolute certainty that Smith's E_8 theory in 8D furnishes three generations of chiral fermions in 4D. Hawking and Pope [65] raised the interesting possibility that there may be a connection between the topology of space–time and the spectrum of elementary particles.

Another interesting project would be also to repeat these calculations for other gauge fields, E_7, E_6, $SO(10)$, $SU(8)$, ..., and compact internal spaces like CP^n, $\mathbf{G}/\mathbf{H}$ coset spaces to find out if instanton configurations trigger a spontaneous compactification to lower dimensions, and if this led to an integer-valued index such that it can accommodate the right number of chiral fermions (three or more generations) in four dimensions. A pure Kaluza–Klein approach was largely abandoned in the 1980s due in part to the realisation by Witten [69] that it was difficult, if not impossible, to obtain chiral fermions from a Kaluza–Klein compactification (of 11D supergravity, in particular) in this way. Dolan and Nash [20] took a different approach to the internal coset spaces $\mathbf{G}/\mathbf{H}$, focusing on the holonomy group, H, rather than G. And, thirdly, one has to verify that all the chiral fermions have precisely the right quantum numbers consistent with the Standard Model and its extensions.

7. On complex geometric domains, couplings, masses, and parameters of the Standard Model

7.1. Evaluation of the coupling constants

By returning to geometric probability methods it was shown [24] that the coupling constants, α_{EM}, α_W, α_C, associated with the EM, weak, and strong (color) force are given by the *ratios* of measures of the sphere S^2 and the Shilov boundaries $Q_3 = S^2 \times RP^1$, *squashed* S^5, respectively, with respect to the Wyler measure $\Omega_{\mathrm{Wyler}}[Q_4]$ of the Shilov boundary $Q_4 = S^3 \times RP^1$ of the polydisc D_4 (eight real dimensions). The latter measure $\Omega_{\mathrm{Wyler}}[Q_4]$ is linked to the geometric coupling strength, α_G, associated with the gravitational force.

The topology of the boundaries (at conformal infinity) of the past and future light cones are spheres S^2 (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. To describe the physics at infinity we will recur to Penrose's ideas [70] of conformal compactifications of Minkowski space–time by attaching the light cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss–Riemann sphere, the conformal group leaves the light cone fixed and does not alter the causal properties of space–time despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski space–time $\bar{M}_4 = (S^3 \times S^1)/Z_2 = S^3 \times RP^1$ is precisely the same as the topology of the Shilov boundary Q_4 of the four complex-dimensional polydisc D_4. The action of the discrete group Z_2 amounts to an antipodal identification of the future null infinity $\mathcal{I}^+$ with the past null infinity $\mathcal{I}^-$; and the antipodal identification of the past time-like infinity i^- with the future time-like infinity, i^+, where the electron emits and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains, G/K [71–73] are not the same as the ordinary topological boundaries (except in some special cases), because the action of the isotropy group, K, of the origin is not necessarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries, which implement the Maldacena–'t Hooft–Susskind *holographic* principle [74] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the polydisc D_4 of four complex dimensions is an eight real-dimensional hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a 4D anti-de Sitter space AdS_4. The polydisc is a Hermitian symmetric homogeneous coset space associated with the 4D conformal group $SO(4, 2)$ since $D_4 = SO(4, 2)/SO(4) \times SO(2)$. Its Shilov boundary Shilov $(D_4) = Q_4$ has precisely the *same* topology as the 4D conformally compactified real Minkowski space–time $Q_4 = \bar{M}_4 = (S^3 \times S^1)/Z_2 = S^3 \times RP^1$. For more details about Shilov boundaries, the conformal group, future tubes, and holography we refer to the article by Gibbons [2, 71, 75].

A typical objection to the possibility of being able to derive the values of the coupling constants, from pure thought alone, is that there are an uncountably infinite number of possible analytical expressions that accurately reproduce the values of the couplings, at any given energy scale, and within the experimental error bounds. However, this is not our case because once the gauge groups $U(1)$, $SU(2)$, and $SU(3)$ are known there are *unique* analytical expressions stemming from geometric probability that furnish the values of the couplings.

Another objection is that it is a meaningless task to try to derive these couplings because these are *not* constants per se but vary with respect to the energy scale. The running of the coupling constants is an *artifact* of the perturbative renormalization group (RG) program. We will see that the values of the couplings derived from geometric probability are precisely those values that correspond to the natural physical scales associated with the EM, weak, and strong forces. The difficulty still remains in explaining *why* this occurs. Namely, why there is a precise correlation among the values of the couplings hereby obtained with the typical energy scales associated with the EM, weak, and strong forces.

Another objection is that physical measurements of irrational numbers are impossible because there are always experimental and physical limitations that rule out the possibility of actually measuring the *infinite* number of digits of an irrational number. Measurements with finite-resolution apparatus are more compatible with *rational* values for the physical constants, rather than irrational numbers. The rational values of physical constants are more amenable to the role of p-adic numbers in physics [76].

This experimental constraint does not exclude the possibility of deriving exact expressions based on π as we shall see. We should not worry about obtaining numerical values within the error bars in the table of the coupling constants because these numbers are based on the values of *other* physical constants; that is, they are based on the particular *consensus* chosen for all of the other physical constants.

In our conventions, $\alpha_{EM} = e^2/4\pi = 1/137.036\ldots$ in the natural units of $\hbar = c = 1$, and the quantities α_{weak}, α_{color} are the geometric probabilities $\tilde{g}_w^2$, $\tilde{g}_c^2$, after *absorbing* the factors of 4π of the conventional $\alpha_w = (g_w^2/4\pi)$, $\alpha_c = (g_c^2/4\pi)$ definitions used in the RG program.

7.2. Evaluation of the fine structure constant

We review work [24] on the derivation of the fine structure constant, the weak and strong coupling, based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of geometric probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy $\Sigma(p)$ as a function of the electron's incoming (outgoing) energy–momentum p_μ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i}{\gamma^\rho(p_\rho - k_\rho) - m}\frac{-ig_{\mu\nu}}{k^2}\gamma^\nu \tag{7.1}$$

The integral is taken with respect to the values of the photon's energy–momentum k^μ. By inspection one can see that the electron self-energy is proportional to the fine structure constant, $\alpha_{EM} \sim e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$), and physically represents the electron's emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy–momentum k_ρ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the geometric probability that an electron emits a photon at $t = -\infty$ (infinite past) and reabsorbs it at a much later time $t = +\infty$ (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ($t = -\infty$) and absorption ($t = +\infty$). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron. The fact that geometric probability is a classical theory does not mean that one cannot derive the fine structure constant (which involves the Planck constant) because the electron self-energy diagram is itself a quantum (one-loop) Feynman process; that is, one can return to geometric probability to assign proper geometrical measures to Feynman diagrams, not unlike the twistor-diagrammatic version of the Feynman rules of quantum field theory.

To define the geometric probability associated with this process of the electron's emission of a photon at i^- ($t = -\infty$), followed by an absorption at i^+ ($t = +\infty$), we must take into account the important fact that the photon is on-shell $k^2 = 0$ *asymptotically* (at $t = \pm\infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region, which is represented by the *interior* of the 4D conformally compactified real Minkowski space–time, which agrees with the Shilov boundary of D_4 (the four-complex-dimensional polydisc) $Q_4 = \bar{M}_4 = (S^3 \times S^1)/Z_2 = S^3 \times RP^1$. Q_4 has four real dimensions, which is half the real dimensions of D_4 ($2 \times 4 = 8$).

The measure associated with the celestial spheres S^2 (associated with the future–past light cones) at time-like infinity i^+, i^-, respectively, is $V(S^2) = 4\pi r^2 = 4\pi (r = 1)$. Thus, the *net* measure corresponding to the two celestial spheres S^2 at time-like infinity $i^\pm$ requires an overall factor of two giving $2V(S^2) = 8\pi(r = 1)$. The factor of $8\pi = 2 \times 4\pi$ can also be interpreted in terms of the two-helicity degrees of freedom, corresponding to a spin-1 massless photon, assigned to the area of the celestial sphere. The geometric probability is defined by the ratio of the (dimensionless volume) measures associated with the celestial spheres S^2 at i^+, i^- time-like infinity, where the photon moves on-shell, relative to the Wyler measure $\Omega_{Wyler}[Q_4]$ associated with the full *interior* region of the conformally compactified 4D Minkowski space $Q_4 = \bar{M}_4 = (S^3 \times S^1)/Z_2 = S^3 \times RP^1$, where the massive electron is confined to move, as it propagates from i^- to i^+, (and *off-shell* photons can also live in)

$$\alpha_{EM} = \frac{2V(S^2)}{\Omega_{Wyler}[Q_4]} = \frac{8\pi}{\Omega_{Wyler}[Q_4]} = \frac{1}{137.036\,08\ldots} \tag{7.2a}$$

after inserting the Wyler measure

$$\Omega_{Wyler}[Q_4] = \frac{V(S^4)V(Q_5)}{[V(D_5)]^{1/4}} = \left(\frac{8\pi^2}{3}\right)\left(\frac{8\pi^3}{3}\right)\left(\frac{\pi^5}{2^4 \times 5!}\right)^{-1/4} \tag{7.2b}$$

The Wyler measure $\Omega_{Wyler}[Q_4]$ [77] is *not* the standard measure (dimensionless volume) $V(Q_4) = 2\pi^3$ calculated by Hua [72] but requires some elaborate procedure.

It was realized by Smith [22] that the presence of the Wyler measure in the expression for α_{EM} given by (2-1) was consistent with Wheeler ideas that the observed values of the coupling constants of the EM, weak, and strong force can be obtained if the geometric force strengths (measures related to volumes of complex homogenous domains associated with the $U(1)$, $SU(2)$, $SU(3)$ groups, respectively) are all *divided* by the geometric force strength of gravity, α_G (related to the $SO(3, 2)$ MMCW gauge theory of gravity), which is not the same as the 4D Newton's gravitational constant $G_N \sim m_{Planck}^{-2}$. Hence, upon dividing these geometric force strengths by the geometric force strength of gravity, α_G, one is dividing by the Wyler measure factor because (as we shall see later) $\alpha_G \equiv \Omega_{Wyler}[Q_4]$.

Furthermore, the expression for $\Omega_{Wyler}[Q_4]$ is also consistent with the Kaluza–Klein compactification procedure of obtaining Maxwell's EM in 4D from *pure* gravity in 5D because Wyler's expression involves a 5D domain D_5 from the very start; that is, to evaluate the Wyler measure $\Omega_{Wyler}[Q_4]$ one requires to embed D_4 into D_5 because the Shilov boundary space $Q_4 = S^3 \times RP^1$ is *not* adequate enough to implement the action of the $SO(5)$ group, the compact version of the anti-de Sitter group $SO(3, 2)$ that is required in the MMCW $SO(3, 2)$ gauge formulation of gravity. However, the Shilov boundary of D_5 given by $Q_5 = S^4 \times RP^1$ is adequate to implement the action of $SO(5)$ via isometries (rotations) on the internal symmetry space $S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4 \to D_5$.

The five complex-dimensional polydisc $D_5 = SO(5, 2)/SO(5) \times SO(2)$ is the 10 real-dimensional hyperboloid $\mathcal{H}^{10}$ corresponding to the relativistic curved phase space of a particle moving in 5D anti-de Sitter space AdS_5. The Shilov boundary Q_5 of D_5 has five real dimensions (half of the 10 real dimensions of D_5). One cannot fail to notice that the hyperboloid $\mathcal{H}^{10}$ can be embedded in the 11-dimensional pseudo-Euclidean $R^{9,2}$ space, with two time-like directions. This is where 11 dimensions sneak into our construction.

Having displayed Wyler's expression of the fine structure constant, α_{EM}, in terms of the ratio of dimensionless measures, we shall present a fiber bundle (a sphere bundle fibration over a complex homogeneous domain) derivation of the Wyler expression based on the bundle $S^4 \to E \to D_5$, and explain below why the propagation (via the determinant of the Feynman propagator) of the electron through the *interior* of the domain D_5 is what accounts for the "obscure" factor $V(D_5)^{1/4}$ in Wyler's formula for α_{EM}.

We begin by explaining why Wyler's measure $\Omega_{Wyler}[Q_4]$ in (7.2) corresponds to the measure of a S^4 bundle fibered over the base curved space $D_5 = SO(5, 2)/SO(5) \times SO(2)$ and *weighted* by a factor of $V(D_5)^{-1/4}$. This $S^4 \to E \to D_5$ bundle is linked to the MMCW $SO(3, 2)$ gauge theory formulation of gravity and explains the essential role of the gravitational interaction of the electron in Wyler's formula corroborating Wheeler's ideas that one must normalize the geometric force strengths with respect to gravity to obtain the coupling constants.

The subgroup $H = SO(5)$ of the isotropy group (at the origin) $K = SO(5) \times SO(2)$ acts naturally on the fibers $F = S^4 = SO(5)/SO(4)$, the internal symmetric space, via isometries (rotations). Locally, and only locally, the fiber bundle E is the product $D_5 \times S^4$. The restriction of the fiber bundle E to the Shilov boundary Q_5 is written as $E|_{Q_5}$ and *locally* is the product of $Q_5 \times S^4$, but this is *not* true globally unless the fiber bundle admits a global section (the bundle is trivial). For this reason the volume $V(E|_{Q_5})$ does not necessary always factorize as $V(Q_5) \times V(S^4)$.

Setting aside this subtlety, we shall pursue a more physical route, already suggested by Wyler in unpublished work [78][1], to explain the origin of the "obscure normalization" factor $V(D_5)^{1/4}$ in Wyler's measure $\Omega_{Wyler}[Q_4] = [V(S^4) \times V(Q_5)/V(D_5)^{1/4}]$, which suggests that the volumes may not factorize. Fn1

The relevant physical feature of this measure factor $V(D_5)^{1/4}$ is that it encodes the *spinorial* degrees of freedom of the electron, like the factor of 8π encodes the two-helicity states of the massless photon. The Feynman propagator of a massive scalar particle (inverse of the Klein–Gordon operator) $(D_\mu D^\mu - m^2)^{-1}$ corresponds to the *kernel* in the Feynman path integral that in turn is associated with the Bergman kernel $K_n(z, z')$ of the complex homogenous domain D_n, which is proportional to the Bergman constant $k_n \equiv 1/V(D_n)$.

$$(D_\mu D^\mu - m^2)^{-1}(x^\mu) = \frac{1}{(2\pi\mu)^D}\int d^Dp \frac{e^{-ip_\mu x^\mu}}{p^2 - m^2 + i\epsilon} \leftrightarrow K_n(z, \bar{z}') = \frac{1}{V(D_n)}(1 - z\bar{z}')^{-2n} \tag{7.3}$$

where we have introduced a momentum scale, μ, to match units in the Feynman propagator expression, and the Bergman kernel $K_n(z, \bar{z}')$ of D_n whose dimensionless entries are $z = (z_1, z_2, \ldots.., z_n)$, $z = (z'_1, z'_2, \ldots, z'_n)$ is given as

$$K_n(z, \bar{z}') = \frac{1}{V(D_n)}(1 - z\bar{z}')^{-2n} \tag{7.4a}$$

where $V(D_n)$ is the dimensionless Euclidean volume found by Hua $V(D_n) = (\pi^n/s^{n-1}n!)$ and satisfies the reproducing and normalization properties

$$f(z) = \int_{D_n} f(\xi)K_n(z, \xi)d^n\xi d^n\bar{\xi} \qquad \int_{D_n} K_n(z, \bar{z})d^nzd^n\bar{z} = 1 \tag{7.4b}$$

The *key* result that can be inferred from the Feynman propagator (kernel) $\leftrightarrow$ Bergman kernel K_n correspondence, when $\mu = 1$, is the $(2\pi)^{-D} \leftrightarrow [V(D_n)]^{-1}$ correspondence; that is, the fundamental hypercell in momentum space $(2\pi)^D$ (when $\mu = 1$) corresponds to the dimensionless volume $V(D_n)$ of the domain, where $D = 2n$ real dimensions. The regularized vacuum-to-vacuum amplitude of a free *real* scalar field is given in terms of the zeta function $\zeta(s) = \sum_i \lambda_i^{-s}$ associated with the eigenvalues of the Klein–Gordon operator by

$$Z = \langle 0|0\rangle = \sqrt{\det\left(D_\mu D^\mu - m^2\right)^{-1}} \sim \exp\left[\frac{1}{2}\frac{d\zeta}{ds}(s = 0)\right] \tag{7.5}$$

[1]We thank F. (Tony) Smith for this information.

In case of a *complex* scalar field we have to *double* the number of degrees of freedom, the amplitude then factorizes into a product and becomes $Z = \det(D_\mu D^\mu - m^2)^{-1}$.

Because the Dirac operator $\mathcal{D} = \gamma^\mu D_\mu + m$ is the "square-root" of the Klein–Gordon operator $\mathcal{D}^\dagger\mathcal{D} = D_\mu D^\mu - m^2 + \mathcal{R}$ ($\mathcal{R}$ is the scalar curvature of space–time that is zero in Minkowski space) we have the numerical correspondence

$$\sqrt{det(\mathcal{D})^{-1}} = \sqrt{\det\left(D_\mu D^\mu - m^2\right)^{-1/2}} = \sqrt{\sqrt{\det\left(D_\mu D^\mu - m^2\right)^{-1}}} \leftrightarrow k_n^{1/4} = \left[\frac{1}{V(D_n)}\right]^{1/4} \tag{7.6}$$

because $\det \mathcal{D}^\dagger = \det \mathcal{D}$, and

$$\det \mathcal{D} = e^{\mathrm{Tr}\ln \mathcal{D}} = e^{\mathrm{Tr}\ln(D_\mu D^\mu - m^2)^{1/2}} = e^{(1/2)\mathrm{Tr}\ln(D_\mu D^\mu - m^2)} = \sqrt{\det\left(D_\mu D^\mu - m^2\right)} \tag{7.7}$$

The vacuum-to-vacuum amplitude of a *complex* Dirac field, Ψ, (a fermion, the electron) is $Z = \det(\gamma^\mu D_\mu + m) = \det \mathcal{D} \sim \exp[-(d\zeta/ds)(s = 0)]$. Notice the $\det(\mathcal{D})$ behavior of the fermion versus the $\det(D_\mu D^\mu - m^2)^{-1}$ behavior of a complex scalar field due to the Grassmanian nature of the Gaussian path integral of the fermions. The vacuum-to-vacuum amplitude of a Majorana (real) spinor (half of the number of degrees of freedom of a complex Dirac spinor) is $Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$. Because the complex Dirac spinor encodes both the dynamics of the electron and its antiparticle, the positron (the negative energy solutions), the vacuum-to-vacuum amplitude corresponding to the electron (positive energy solutions, propagating forward in time) must be then $Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$.

Therefore, the origin of the "obscure" factor $V(D_5)^{1/4}$ in Wyler's formula is the *normalization* condition of $V(S^4) \times V(Q_5)$ by a factor of $V(D_5)^{1/4}$ stemming from the correspondence $V(D_5)^{1/4} \leftrightarrow Z = \sqrt{\det(\gamma^\mu D_\mu + m)}$ and which originates from the vacuum-to-vacuum amplitude of the fermion (electron) as it propagates forward in time in the domain D_5. These last relations emerge from the correspondence between the Feynman fermion (electron) propagator in Minkowski space–time and the Bergman kernel of the complex homogenous domain after performing the Wyler map between an unbounded domain (the interior of the future light cone of space–time) to a bounded one. In general, the Bergman kernel gives rise to a Kahler potential $F(z, \bar{z}) = \log K(z, \bar{z})$ in terms of which the Bergman metric on D_n is given by

$$g_{i\bar{j}} = \frac{\partial^2 F}{\partial z^i \partial \bar{z}^i} \tag{7.8}$$

We must emphasize that this geometric probability explanation *is very different* from the interpretations provided in refs. 77 and 79 and properly accounts for all the numerical factors. Concluding, the geometric probability that an electron emits a photon at $t = -\infty$ and absorbs it at $t = \pm\infty$, is given by the *ratio* of the dimensionless measures (volumes)

$$\alpha_{\mathrm{EM}} = \frac{2V(S^2)}{\Omega_{\mathrm{Wyler}}[Q_4]} = (8\pi)\frac{1}{V(S^4)}\frac{1}{V(Q_5)}[V(D_5)]^{1/4} = \frac{9}{8\pi^4}\left(\frac{\pi^5}{2^4 \times 5!}\right)^{1/4} = \frac{1}{137.036\,08\ldots} \tag{7.9}$$

in very good agreement with the experimental value. This is easily verified after one inserts the values of the Euclideanized *regularized* volumes found by Hua [72]

$$V(D_5) = \frac{\pi^5}{2^4 \times 5!} \qquad V(Q_5) = \frac{8\pi^3}{3} \qquad V(S^4) = \frac{8\pi^2}{3} \tag{7.10}$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1} n!} \qquad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{7.11}$$

$$V(Q_n) = V(S^{n-1} \times RP^1) = V(S^{n-1}) \times V(RP^1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)} \tag{7.12}$$

Objections were raised to Wyler's original expression by Robertson [80]. One of them was that the hyperboloids (discs) are *not* compact and their volumes diverge because the Lobachevsky metric diverges on the boundaries of the polydiscs. Gilmore explained [80] why it is required that one uses the Euclideanized regularized volumes because Wyler showed that it is possible to map an unbounded physical domain (the interior of the future light cone) onto the interior of a homogenous bounded domain without losing the causal structure and on which there exists also a complex structure. A study of Shilov boundaries, holography, and the future tube can be found in ref. 75.

Furthermore, to resolve the scaling problems of Wyler's expression raised by Robertson, Gilmore showed why it is essential to use *dimensionless* volumes by setting the throat sizes of the anti-de Sitter hyperboloids to $r = 1$, because this is the only choice for r where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactory solved by Gilmore [80]. Thus, all the volumes in this section, and in the next sections, are based on setting the scaling factor $r = 1$.

The question as to *why* the value of α_{EM} obtained in Wyler's formula is precisely the value of α_{EM} observed at the *scale* of the Bohr radius, a_{B}, has not been solved, to our knowledge. The Bohr radius is associated with the ground (most stable) state of the hydrogen atom. The spectrum generating group of the hydrogen atom is well known to be the conformal group $SO(4, 2)$ because there are two conserved vectors: the angular momentum and the Runge–Lentz vector. After quantization, one has two commuting $SU(2)$ copies $SO(4) = SU(2) \times SU(2)$. Thus, it makes physical sense that the Bohr scale should appear in this construction.

Bars has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes, and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of M, F, S theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036 \times 2 \times 13.6$ eV $\sim 3.72 \times 10^3$ eV. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of the electron, and the classical electron radius are all related to each other by a successive scaling in products of α_{EM}.

To finalize this section and based on the MMCW $SO(3, 2)$ gauge theory formulation of gravity, with a Gauss–Bonnet topological term plus a cosmological constant, the (dimensionless) Wyler measure was *defined* as the geometric coupling strength of gravity [22]

$$\Omega_{\mathrm{Wyler}}[Q_4] = \frac{V(S^4)V(Q_5)}{[V(D_5)]^{1/4}} \equiv \alpha_{\mathrm{G}} \tag{7.13}$$

The relationship between α_G and the Newtonian gravitational G constant is based on the value of the coupling ($1/16\pi G$) appearing in the Einstein–Hilbert Lagrangian ($R/16\pi G$), and goes as follows:

$$(16\pi G)\left(m^2_{\text{Planck}}\right) = \alpha_{\text{EM}}\alpha_G = 8\pi \Rightarrow G = \frac{1}{16\pi}\frac{8\pi}{m^2_{\text{Planck}}} = \frac{1}{2m^2_{\text{Planck}}}$$

$$\Rightarrow Gm^2_{\text{proton}} = \frac{1}{2}\left(\frac{m_{\text{proton}}}{m_{\text{Planck}}}\right)^2 \sim 5.9 \times 10^{-39} \qquad (7.14)$$

and in natural units $\hbar = c = 1$ yields the physical force strength of gravity at the Planck energy scale 1.22×10^{19} GeV. The Planck mass is obtained by equating the Schwarzschild radius $2Gm_{\text{Planck}}$ to the Compton wavelength $1/m_{\text{Planck}}$ associated with the mass; where $m_{\text{Planck}}\sqrt{2} = 1.22 \times 10^{19}$ GeV and the proton mass is 0.938 GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{\text{Planck}} = 1.22 \times 10^{19}$ GeV.

7.3. Evaluation of the weak and strong couplings

We turn now to the derivation of the other coupling constants. The fiber bundle picture of the previous section is essential in our construction. The weak and strong geometric coupling constant strengths, defined as the probability for a particle to emit and later absorb $SU(2)$ and $SU(3)$ gauge bosons, respectively, can both be obtained by using the main formula derived from geometric probability (as ratios of dimensionless measures and volumes) after one identifies the suitable homogeneous domains and their Shilov boundaries to work with.

Because massless gauge bosons live on the light cone, a null boundary in Minkowski space–time, upon performing the Wyler map, the gauge bosons are confined to live on the Shilov boundary. Because the $SU(2)$ bosons $W^{\pm}$, Z^0, and the eight $SU(3)$ gluons have *internal* degrees of freedom (they carry weak and color charges) one must also include the measure associated with the their respective internal spaces; namely, the measures relevant to geometric probability calculations are the measures corresponding to the appropriate sphere bundles fibrations defined over the complex bounded homogenous domains $S^m \to E \to \mathcal{D}_n$.

Furthermore, the geometric probability interpretation for α_{weak}, α_{strong} agrees with Wheeler's ideas [22] that one must normalize these geometric force strengths with respect to the geometric force strength of gravity $\alpha_G = \Omega_{\text{Wyler}}[Q_4]$ found in the last section. Hence, after these explanations, we will show why the weak and strong couplings are given, respectively, by the *ratio* of the measures (dimensionless volumes):

$$\alpha_{\text{weak}} = \frac{\Omega[Q_3]}{\Omega_{\text{Wyler}}[Q_4]} = \frac{\Omega[Q_3]}{\alpha_G} = \frac{\Omega[Q_3]}{(8\pi/\alpha_{\text{EM}})} \qquad (7.15)$$

$$\alpha_{\text{color}} = \frac{\Omega[\text{squashed } S^5]}{\Omega_{\text{Wyler}}[Q_4]} = \frac{\Omega[\text{squashed } S^5]}{\alpha_G} = \frac{\Omega[\text{squashed } S^5]}{(8\pi/\alpha_{\text{EM}})} \qquad (7.16)$$

As always, one must insert the values of the regularized (Euclideanized) dimensionless volumes provided by Hua [72] (set the scale $r = 1$). We must also clarify and emphasize that we define the quantities α_{weak} and α_{color} as the probabilities $\tilde{g}^2_w$, $\tilde{g}^2_c$, by absorbing the factors of 4π in the conventional $\alpha_w = (g^2_w/4\pi)$, $\alpha_c = (g^2_c/4\pi)$ definitions (based on the RG program) into our definitions of probability $\tilde{g}^2_w$, $\tilde{g}^2_c$.

Let us evaluate the α_{weak}. The internal symmetry space is $CP^1 = SU(2)/U(1)$ (a sphere $S^2 \sim CP^1$) where the isospin group $SU(2)$ acts via isometries on CP^1. The Shilov boundary of D_2 is $Q_2 = S^1 \times RP^1$ but is not adequate to accommodate the action of the isospin group $SU(2)$. One requires the Shilov boundary of D_3 given by $Q_3 = S^2 \times S^1/Z_2 = S^2 \times RP^1$ that can accommodate the action of the $SU(2)$ group on S^2. A fiber bundle over $D_3 = SO(3, 2)/SO(3) \times SO(2)$ whose $H = SO(3) \sim SU(2)$ subgroup of the isotropy group (at the origin) $K = SO(3) \times SO(2)$ acts on S^2 by simple rotations. Thus, the relevant measure is related to the fiber bundle E restricted to Q_3 and is written as $V(E|_{Q_3})$.

One must notice that because the $SU(2)$ group is a double-cover of $SO(3)$, as one goes from the $SO(3)$ action on S^2 to the $SU(2)$ action on S^2, one must take into account an extra factor of two giving then

$$V(CP^1) = V\left[\frac{SU(2)}{U(1)}\right] = 2V\left[\frac{SO(3)}{U(1)}\right] = 2V(S^2) = 8\pi \qquad (7.17)$$

To obtain the weak coupling constant due to the exchange of $W^{\pm}$ Z^0 bosons in the four-point tree-level processes involving four leptons, like the electron, muon, tau, and their corresponding neutrinos (leptons are fundamental particles that are lighter than mesons and baryons), which are confined to move in the interior of the domain D_3, and can emit (absorb) $SU(2)$ gauge bosons, $W^{\pm}$ Z^0, in the respective s, t, u channels, one must take into account a factor of the square root of the determinant of the fermionic propagator, $\sqrt{\det \mathcal{D}^{-1}} = \sqrt{\det(\gamma^\mu D_\mu + m)^{-1}}$, for each *pair* of leptons, as we did in the previous section when an electron emitted and absorbed a photon. Because there are *two* pairs of leptons in these four-point tree-level processes involving *four* leptons, one requires *two* factors of $\sqrt{\det(\gamma^\mu D_\mu + m)^{-1}}$, giving a net factor of $\det(\gamma^\mu D_\mu + m)^{-1}$ and which corresponds now to a net normalization factor of $k_n^{1/2} = [1/V(D_3)]^{1/2}$, after implementing the Feynman kernel ↔ Bergman kernel correspondence. Therefore, after taking into account the result of (7.17), the measure of the $S^2 \to E \to D_3$ bundle, restricted to the Shilov boundary Q_3, and weighted by the net normalization factor $[1/V(D_3)]^{1/2}$, is

$$\Omega(Q^3) = 2V(S^2)\frac{V(Q_3)}{V(D_3)^{1/2}} \qquad (7.18)$$

Therefore, the geometric probability expression is given by the ratio of measures (dimensionless volumes)

$$\alpha_{\text{weak}} = \frac{\Omega[Q^3]}{\Omega_{\text{Wyler}}[Q_4]} = \frac{\Omega[Q^3]}{\alpha_G} = \frac{2V(S^2)V(Q_3)}{V(D_3)^{1/2}}\frac{\alpha_{\text{EM}}}{8\pi}$$

$$= (8\pi)(4\pi^2)\left(\frac{\pi^3}{24}\right)^{-1/2}\frac{\alpha_{\text{EM}}}{8\pi} = 0.2536\ldots \qquad (7.19)$$

that corresponds to the weak coupling constant ($g^2/4\pi$ based on the RG convention) at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M^2_{W_+} + M^2_{W_-} + M^2_Z} \qquad (7.20)$$

after we have inserted the expressions (setting the scale $r = 1$)

$$V(S^2) = 4\pi \qquad V(Q_3) = 4\pi^2 \qquad V(D_3) = \frac{\pi^3}{24} \qquad (7.21)$$

into (7.19). The relationship to the Fermi coupling G_{Fermi} goes as follows (after setting the energy scale $E = M = 146$ GeV):

$$G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F m^2_{\text{proton}} = \left(\frac{\alpha_W}{M^2}\right)m^2_{\text{proton}} = 0.2536$$

$$\times \left(\frac{m_{\text{proton}}}{146 \text{ GeV}}\right)^2 \sim 1.04 \times 10^{-5} \qquad (7.22)$$

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in very good agreement with experimental observations. Once more, it is unknown why the value of α_{weak} obtained from geometric probability corresponds to the energy scale related to the W_+, W_-, Z_0 boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of α_{color} from (7.16) after one defines the suitable fiber bundle. The calculation is based on the book by L.K. Hua [72, pp. 40 and 93]. The symmetric space with the $SU(3)$ color force as a local group is $SU(4)/SU(3) \times U(1)$, which corresponds to a bounded symmmetric domain of type $I(1, 3)$ and has a Shilov boundary that Hua calls the "characteristic manifold" $CI(1, 3)$. The volume $V[CI(m, n)]$ is

$$V(CI) = \frac{(2\pi)^{mn-m(m-1)/2}}{(n-m)!(n-m+1)!\ldots(n-1)!} \tag{7.23}$$

so that for $m = 1$ and $n = 3$ the relevant volume is then $V(CI) = (2\pi)^3/2! = 4\pi^3$. We must remark at this point that $CI(1, 3)$ is *not* the standard round S^5 but is the *squashed* five-dimensional $\tilde{S}^5$[2].

The domain of which $CI(1, 3)$ is the Shilov boundary is denoted by Hua as $RI(1, 3)$ and whose volume is

$$V(RI) = \frac{1!2!\ldots(m-1)!1!2!\ldots(n-1)!\pi^{mn}}{1!2!\ldots(m+n-1)!} \tag{7.24}$$

so that for $m = 1$ and $n = 3$ it gives $V(RI) = 1!2!\pi^3/1!2!3! = \pi^3/6$ and it also agrees with the volume of the standard six-ball.

The internal symmetry space (fibers) is $CP^2 = SU(3)/U(2)$ whose isometry group is the color $SU(3)$ group. The base space is the 6D domain $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$ whose subgroup $SU(3)$ of the isotropy group (at the origin) $K = SU(3) \times U(1)$ acts on the internal symmetry space CP^2 via isometries. In this special case, the Shilov and ordinary topological boundary of B_6 both coincide with the *squashed* S^5 [22].

Because Gilmore, in response to Robertson's objections to Wyler's formula [77], has shown that one must set the scale $r = 1$ of the hyperboloids $\mathcal{H}^n$ (and S^n) and use *dimensionless* volumes (if we were to equate the volumes $V(CP^2) = V(S^4, r = 1)$ [22]) this would be tantamount of choosing another scale [81] R (the unit of geodesic distance in CP^2) that is *different* from the unit of geodesic distance in S^4 when the radius $r = 1$, as required by Gilmore. Hence, a bundle map $E \to E'$ from the bundle $CP^2 \to E \to B_6$ to the bundle $S^4 \to E' \to B_6$, would be required that would allow us to replace the $V(CP^2)$ for $V(S^4, r = 1)$. Unless one decides to *calibrate* the unit of geodesic distance in CP^2 by choosing $V(CP^2) = V(S^4)$.

Using again the same results described after (6.2), because a quark can emit and absorb later on a $SU(3)$ gluon (in a one-loop process), and is confined to move in the interior of the domain B_6, there is *one* factor only of the square root of the determinant of the Dirac propagator $\sqrt{\det \mathcal{D}^{-1}} = \sqrt{\sqrt{\det(D_\mu D^\mu - m^2)^{-1}}}$ and which is associated with a normalization factor of $k_n^{1/4} = [1/V(B_6)]^{1/4}$. Therefore, the measure of the bundle $S^4 \to E' \to B_6$, restricted to the *squashed* S^5 (Shilov boundary of B^6) and weighted by the normalization factor $[1/V(B_6)]^{1/4}$, is then

$$\Omega[\text{squashed } S^5] = \frac{V(S^4)V(\text{squashed } S^5)}{V(B_6)^{1/4}} \tag{7.25}$$

and the ratio of measures

$$\begin{aligned}\alpha_s &= \frac{\Omega[\text{squashed } S^5]}{\Omega_{\text{Wyler}}[Q_4]} = \frac{\Omega[\text{squashed } S^5]}{\alpha_G} \\ &= \frac{V(S^4)V(\text{squashed } S^5)}{V(B_6)^{1/4}}\frac{\alpha_{\text{EM}}}{8\pi} = \left(\frac{8\pi^2}{3}\right)(4\pi^3)\left(\frac{\pi^3}{6}\right)^{-1/4}\frac{\alpha_{\text{EM}}}{8\pi} \\ &= 0.6286\ldots\end{aligned} \tag{7.26}$$

matches, remarkably, the strong coupling value $\alpha_s = g^2/4\pi$ at an energy E related precisely to the pion masses [22]

$$E = 241\text{ MeV} = 0.241\text{ GeV} \sim \sqrt{m_{\pi^+}^2 + m_{\pi^-}^2 + m_{\pi^0}^2} \tag{7.27}$$

The one-loop RG flow of the coupling is given by

$$\alpha_s(E^2) = \alpha_s(E_0^2)\left\{1 + \frac{[11 - (2/3)N_f(E^2)]}{4\pi}\alpha_s(E_0^2)\ln\left(\frac{E^2}{E_0^2}\right)\right\}^{-1} \tag{7.28}$$

where $N_f(E^2)$ is the number of quark flavors whose mass $M^2 < E^2$. For the specific numerical details of the evaluation (in energy intervals given by the diverse quark masses) of the RG flow equation, (7.28), that yields $\alpha_s(E = 241\text{ MeV}) \sim 0.6286$ we refer to [22]. Once more, it is unknown why the value of α_{color} obtained from geometric probability corresponds to the energy scale $E = 241$ MeV related to the masses of the pions. The pions are the known lightest quark–antiquark pairs that feel the strong interaction.

Rigorously speaking, one should include higher-loop corrections to (7.28) as shown by Weinberg [82] to determine the values of the strong coupling at energy scales $E = 241$ MeV. This issue and the subtleties behind the calibration of scales (volumes) by imposing the condition $V(CP^2) = V(S^4)$ need to be investigated. For example, one could calibrate lengths in terms of the units of geodesic distance in CP^2 (based on Gilmore's choice of $r = 1$) giving $V(CP^2) = V(S^5; r = 1)/V(S^1; r = 1) = \pi^2/2!$ [81], and it leads now to the value of $\alpha_s = 0.117\,862\,5$, which is very close to the value of α_s at the energy scale of the Z boson mass (91.2 GeV) and given by $\alpha_s = 0.118$.

7.4. Evaluation of particle masses

In this subsection we will review closely the derivation of the particle masses by Smith [22, 23] and add a few results based on the work by Gonzalez–Martin [25].

7.4.1. The electroweak bosons

The triplet (W^+, W^-, Z) couples directly with the Higgs scalar, which carries the Higgs mechanism by which W_0 becomes the physical Z, so that the total mass of the triplet (W^+, W^-, Z) is equal to the VEV, v, of the Higgs scalar field, $v = 252.514$ GeV.

To find the individual masses of members of the triplet (W^+, W^-, Z), look at the triplet (W^+, W^-, Z), which can be represented by the 3-sphere S^3. The Hopf fibration of S^3 as $S^1 \to S^3 \to S^2$ gives a decomposition of the W bosons into the neutral W_0 corresponding to S^1 and the charged pair W^+ and W^- corresponding to S^2. The mass ratio of the sum of the masses of W^+ and W^- to the mass of W_0 should be the volume ratio of the S^2 in S^3 to the S^1 in S^3.

The unit sphere S^3 in R^4 is normalized by 1/2. The unit sphere S^2 in R^3 is normalized by $1/\sqrt{3}$. The unit sphere S^1 in R^2 is normalized by $1/\sqrt{2}$. The ratio of the sum of the W^+ and W^- masses to the W_0 (Z) mass should then be $(2/\sqrt{3})V(S^2)/(2/\sqrt{2})V(S^1) = 1.632\,993$.

Because the total mass of the triplet (W^+, W^-, Z) is 259.031 GeV, and the charged weak bosons have equal mass, we can infer from the prior mass ratio $1.632\,993 = 2M_{W\pm}/M(Z)$, that $M_{W^+} = M_{W^-} = 80.327$ GeV; $M_Z = 98.38$ GeV. Radiative corrections are not taken

[2]F. (Tony) Smith, private communication.

into account here, and may change these tree-level values somewhat.

7.4.2. The Higgs mass: Φ_0, Φ^+.

As with force strengths, the calculations produce ratios of masses, so that only one mass needs to be chosen to set the mass scale. In the unitary gauge of the Standard Model [82], after a $SU(2) \times U(1)$ gauge transformation, the charged component of the complex scalar Higgs doublet $\Phi^{(+)}$ is gauged to zero, and the neutral one $\Phi^{(0)}$ is Hermitian with a positive VEV $\langle \Phi^{(0)} \rangle = v$. In Smith's model, the value of the fundamental mass scale VEV, v of the Higgs scalar field was set to be equal to the sum of the physical masses of the weak bosons, W_+, W_-, Z. The electron mass is the only parameter input by hand and set to be 0.5110 MeV.

The relationship between the Higgs mass and v is given by the Ginzburg–Landau term from the Mayer–Trautman mechanism [63]. The authors [64] found that the invariant meaning of the self-coupling λ of the quartic Higgs terms is nothing but the ratio of two mass scales: $\lambda = 3(M_H/\langle \Phi^{(0)} \rangle)^2$. The idea of the top quark condensate [83] explains naturally the large top mass of the order of the electroweak symmetry breaking scale. In the explicit formulation of this idea, often called the "top mode standard model", the scalar bound state of $\bar{t}t$ plays the role of the Higgs boson in the Standard Model.

In Smith's 8D model the Higgs has also the structure of a top quark condensate $\bar{t}t$ in which a Higgs located at a point in the 4D space–time is connected to a $\bar{t}t$ condensate in the internal four-dimensional space CP^2 in such a way that the three vertices of the Higgs-$\bar{t}t$ system are connected by three lines forming an equilateral triangle. Because of the equilateral triangle configuration of these lines, Smith argues that the self-coupling λ constant of the Higgs quartic coupling $\lambda\Phi^4$ should contain a trigonometric reduction factor associated with a $\pi/6$ angle projection onto the 4D space–time so that now the value $\lambda = 1$ should be $\lambda = [\cos(\pi/6)]^2 = (0.866)^2$. The square is due to the combination $\Phi^4 = (\text{Higgs} \cdot \bar{t}t)^2$. Such value, according to Smith, is consistent with the Higgs–top quark condensate model of Hashimoto et al. [84] where the Standard Model gauge bosons and the third generation of quarks and leptons are put in higher D (= 6, 8, 10, …) dimensions. They find that the top quark condensate can be the maximal attractive channel for $D = 8$.

Therefore, by including this extra reduction factor, according to Smith, the Higgs mass becomes

$$m_H = v\frac{\cos(\pi/6)}{\sqrt{3}} = 126.257 \text{ GeV} \tag{7.29}$$

which agrees with the effective Higgs mass observed by the Large Hadron Collider.

7.4.3. The leptons' and quarks' masses

Gonzalez–Martin [25], in a geometric approach to the lepton and meson masses, which was based on the volumes of complex homogeneous domains, recurred to the cosets

$$K = \frac{SL(4,R)}{SL(2,C) \times SO(2)} \simeq \frac{SO(3,3)}{SL(2,C) \times SO(2)} \qquad C = \frac{Sp(4,R)}{Sp(2,C)} \simeq \frac{SO(3,2)}{SO(3,1)} \tag{7.30}$$

and the Lorentz boost integrals

$$I_K(\beta) = \int_0^\beta \sinh^3 \beta d\beta \qquad I_C(\beta) = \int_0^\beta \sinh^2 \beta d\beta \tag{7.31}$$

to extract the *finite* parts of the infinite volumes of the noncompact coset spaces K, and C after dividing their infinite values by the Lorentz boost integrals as follows:

$$V(K)_{\text{finite}} = \frac{V(K)}{I_K(\beta)} = \frac{2^5\pi^6 I_K(\beta)}{I_K(\beta)} = 2^5\pi^6 \tag{7.32}$$

$$V(C)_{\text{finite}} = \frac{V(C)}{I_C(\beta)} = \frac{(16\pi/3)I_C(\beta)}{I_C(\beta)} = \frac{16\pi}{3} \tag{7.33}$$

The ratio of the finite parts of the volumes yields the proton to electron mass ratio

$$\frac{V(K)_{\text{finite}}}{V(C)_{\text{finite}}} = 6\pi^5 = 1836.1181 \sim \frac{m_{\text{proton}}}{m_{\text{electron}}} \tag{7.34}$$

After taking families of topological excitations corresponding to mappings of n-spheres S^n to the group space, Gonzalez–Martin [25] found that the mass of certain leptons is proportional to integer powers of the volume $V(C_n)$, which depends on the wrapping number, n, as

$$\begin{aligned} V(C_n) &= V[U(1)][V(C)_{\text{finite}}]^{n+1} = 4\pi\left(\frac{16\pi}{3}\right)^{n+1} \qquad n \neq 0 \\ V(C)_{\text{finite}} &= \frac{16\pi}{3} \end{aligned} \tag{7.35}$$

The bare mass of the trivial excitation $n = 0$ is taken to be related to the electron mass and is proportional to the volume $V(C)_{\text{finite}} = 16\pi/3$ so that the masses for other values of n are

$$m_n = m_e 4\pi\left(\frac{16\pi}{3}\right)^n \qquad 0 < n \leq 2 \tag{7.36}$$

When $n = 1$ and $m_e = 0.511$ MeV, the theoretical results give 107.5916 MeV for the muon mass. For $n = 2$ they give 1770.3 MeV for the tau mass. Using the additional geometric interaction energy in a muon–neutrino system, the main leptonic mass contribution to the pion and kaon masses are calculated to be, respectively, 140.88 and 494.76 MeV.

Smith [22, 23] takes the spinor fermion volume to be the Shilov boundary corresponding to the same symmetric space on which $Spin(8)$ acts as a local gauge group that is used to construct 8-dimensional vector space–time: the symmetric space $Spin(10)/Spin(8) \times U(1)$ corresponds to a bounded Hua domain of type $IV8$ whose Shilov boundary is $RP^1 \times S^7$. Smith normalizes the volume V(electron) to 1. To obtain the proton mass, comprised of two up quarks and a down quark, Smith inserted the volume of the domain $IV8$ to be $\pi^5/3$; included a quark-gravity enhanced extra contribution by a factor of six (three colors and three anticolors); and an extra factor of three (based in setting the constituent masses of the up and down quarks to be equal so that $m_u = m_d = m_{\text{proton}}/3$) so that Smith [23] gets the proton-to-electron mass ratio to be $6 \times (\pi^5/3) \times 3 = 6\pi^5$, which is the same ratio value obtained by Gonzalez–Martin [25] earlier. This proton-to-electron mass ratio [25] was known to Wyler, Lenz, and Good [85].

Therefore, the proton mass obtained by both authors is $6\pi^5 m_e = 6\pi^5 \times 0.5110$ MeV $= 938.25$ MeV, which is close to the experimental value of 938.27 MeV. The proton mass is calculated as the sum of the constituent masses of its constituent quarks $m_{\text{proton}} = m_u + m_u + m_d = 938.25$ MeV. The constituent masses of the up and down quark are then $m_u = m_d = 2\pi^5 m_e = 312.75$ MeV.

Because quarks are confined, unobserved, the constituent masses must not be confused with the current masses listed in the particle data booklet and defined in a mass-independent subtraction scheme at a scale of the order of 2 GeV. A constituent quark is a current quark with a covering [86]. In the low energy limit of QCD, a description by means of perturbation theory is not possible. According to the Feynman diagrams, constituent quarks seem to be "dressed" current quarks, that is, current quarks surrounded by a cloud of virtual quarks and gluons. This cloud in the end explains the large constituent-quark masses.

Fermion masses are calculated in ref. 23 as a product of four factors: $V(Q_{\text{fermion}}) \times N(\text{gravity}) \times N(\text{octonion}) \times N(\text{symmetry})$, where $V(Q_{\text{fermion}})$ is the volume of the part of the Weyl (half-spinor) fermion particle manifold $S^7 \times RP^1$ that is related to the fermion particle by photon, weak boson, and gluon interactions; N(gravity) is a gravity enhacement factor; and N(octonion) is an octonion number factor relating the up-type quark to the down-type quark in each generation beyond the first one. The N(octonion) number is set to unity for the first generation. N(symmetry) is an internal symmetry factor relating the second and third generation massive leptons to the first generation fermions.

Here is a summary of the results of calculations of tree-level fermion masses (quark masses are constituent masses) obtained by Smith [23]. One may compare these values with the ones listed in ref. 86.

The neutrino masses are set to zero at the tree level. Taking the electron mass to be m_e = 0.5110 MeV, the other values for the masses are obtained in relation to the electron mass giving: m_{muon} = 104.8 MeV, and m_{tau} = 1.88 GeV. The constituent masses of the quarks are $m_d = m_u$ = 312.8 MeV; m_s = 625 MeV; m_c = 2.09 GeV; m_b = 5.63 GeV; and the top quark m_t = 130 GeV. The controversy with the establishment result value for the top (truth) quark mass of 174.2 ± 3.3 GeV is because Smith [22, 23] believes that the Fermilab figure is incorrect because it is based on an analysis of semileptonic events, does not handle background correctly, and ignores signals that are in rough agreement with tree-level constituent mass values close to 130 GeV. The combinatorics and more details about the fermion mass calculations can be found in ref. 23.

7.5. Cabibbo–Kobayashi–Maskawa parameters and neutrinos

In the Standard Model of particle physics, the Cabibbo–Kobayashi–Maskawa matrix (CKM matrix, quark mixing matrix, sometimes also called KM matrix) is a unitary matrix [87] that contains information on the strength of flavour-changing weak decays. Technically, it specifies the mismatch of quantum states of quarks when they propagate freely and when they take part in weak interactions. It is important in the understanding of CP violation.

Smith [23] used the following formulas based on the preceding masses to calculate the Cabibbo–Kobayashi–Maskawa parameters

$$\sin(\theta_{12}) = s_{12} = \frac{me + 3md + 3mu}{\sqrt{\left(m_e^2 + 3m_d^2 + 3m_u^2\right) + \left(m_\mu^2 + 3m_s^2 + 3m_c^2\right)}} = 0.222\,198 \tag{7.37a}$$

$$\sin(\theta_{13}) = s_{13} = \frac{m_e + 3m_d + 3m_u}{\sqrt{\left(m_e^2 + 3m_d^2 + 3m_u^2\right) + \left(m_\tau^2 + 3m_b^2 + 3m_t^2\right)}} = 0.004\,608 \tag{7.37b}$$

$$\sin(\tilde{\theta}_{23}) = \frac{m_\mu + 3m_s + 3m_c}{\sqrt{\left(m_\tau^2 + 3m_b^2 + 3m_t^2\right) + \left(m_\mu^2 + 3m_s^2 + 3m_c^2\right)}} \tag{7.37c}$$

$$\sin(\theta_{23}) = s_{23} = \sin(\tilde{\theta}_{23})\sqrt{\frac{\sum_{f,2}}{\sum_{f,2}}} = 0.042\,348\,86 \tag{7.37d}$$

where $\sum_{f,2}$ and $\sum_{f,1}$ are the sums over the second and first generation masses, respectively. The CP-violating phase angle used by Smith is δ_{13} = 70.529°. We may compare these values in (7.37) with the currently best known values for the standard parameters [87]: θ_{12} = 13.04° ⇒ $\sin(\theta_{12})$ = 0.225 631; θ_{13} = 0.201° ⇒ $\sin(\theta_{13})$ = 0.003 508; θ_{23} = 2.38° ⇒ $\sin(\theta_{23})$ = 0.041 526 and one finds close agreement with the numbers in (7.37). The CP violating phase is δ_{13} = 1.20 rad = 68.7549° is also close to the CP-violating phase angle δ_{13} = 70.529° in refs. 22 and 23.

The neutrino masses were zero at tree level in Smith's model. They receive loop corrections. The heaviest neutrino mass state, ν_3, corresponds to a neutrino whose propagation begins and ends in the CP^2 internal symmetry space, lying entirely therein. The results by Smith [23] are

$$M_{\nu_3} = \sqrt{2}m_e G_{\text{weak}} m_{\text{proton}}^2 \alpha_E == 1.4 \times 5 \times 10^5 \times 1.05 \times 10^{-5} \times \left(\frac{1}{137}\right)\text{eV} = 5.4 \times 10^{-2}\ \text{eV} \tag{7.38}$$

The intermediate mass state, ν_2, corresponds to a neutrino whose propagation begins in CP^2 and ends in the physical Minkowski space, or vice versa. The first-order corrected mass of ν_2 is $M_{\nu_2} = M_{\nu_3}/\text{vol}(CP^2) = 5.4 \times 10^{-2}/6 = 9 \times 10^{-3}$ eV.

The low mass state, ν_1, corresponds to a neutrino whose propagation begins and ends in physical Minkowski space–time. The first-order corrected mass of ν_1 is $M_{\nu_1} = M_{\nu_2}/\text{vol}(CP^2) = 9 \times 10^{-3}/6 = 1.5 \times 10^{-3}$ eV.

The neutrino mixing matrix calculation was based in using the Stella octangula configuration of two dual tetrahedra. This is because the neutrino mixing matrix has a three-generation structure so it has the same phase structure as the Cabibo–Kobayashi–Mawakaw quark mixing matrix. The unitarity triangle angles found by Smith are: $\beta = \arccos(2\sqrt{2}/3) = 19.471\,220°$; α = 90°, and $\gamma = \arcsin(2\sqrt{2}/3) = 70.528\,779°$.

In particle physics, the Pontecorvo–Maki–Nakagawa–Sakata matrix[88], lepton mixing matrix, or neutrino mixing matrix, is a unitary matrix that contains information on the mismatch of quantum states of leptons when they propagate freely and when they take part in the weak interactions. It is important in the understanding of neutrino oscillations.

Experimentally, the mixing angles were established to be approximately Θ_{12} = 34°, Θ_{23} = 45°, and Θ_{13} = 9.1° (as of 3 April 2013) [88]. Smith' s convention for the angles differs by an extra factor of two so $2\Theta_{12}$ = 64° is close to the values of γ; $2\Theta_{23}$ = 90° agrees with the value of α; and $2\Theta_{13}$ = 18.2° is close to his value of β. We refer to ref. 23 for explicit details.

7.6. Other approaches to obtain the physical constants

Beck [26] obtained all of the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dimensional lattices (two-dimensional nonlinear sigma-like models that resemble Feynman's chess-board lattice models) based on stochastic quantization methods. The results by Smith [22] (also based on Feynman's chess board models and hyperdiamond lattices) are analytical rather than being numerical [26] and it is not clear if there is any relationship between these latter two approaches. Noyes proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for *certain* values of p = primes [89], and obtained a quite large number of satisfactory values for the physical parameters. An interesting coincidence is related to the iterated Mersenne prime sequence

$$\begin{aligned} &M_2 = 2^2 - 1 = 3 \qquad M_3 = 2^3 - 1 = 7 \\ &M_7 = 2^7 - 1 = 127 \qquad 3 + 7 + 127 = 137 \\ &M_{127} = 2^{127} - 1 \sim 1.69 \times 10^{38} \sim \left(\frac{M_{\text{Planck}}}{m_{\text{proton}}}\right)^2 \end{aligned} \tag{7.39}$$

Pitkanen also developed methods to calculate physical masses returning to a p-adic hierarchy of scales based on Mersenne primes [90].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in ref. 91. These are numbers that can be written in terms of sums of their divisors, including unity, like 6 = 1 + 2 + 3, and are of the form $P(p) = (1/2)\, 2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of p = prime yield primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The number of generators of the anomaly-free groups $SO(32)$, $E_8 \times E_8$ of the 10-dimensional superstring is 496, which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even a perfect number because $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in physics and string theory see ref. 92.

A lot more work needs to be done to be able to answer the question: is all this just a mere numerical coincidence or is it design? However, the results of the previous sections indicate that it is very *unlikely* that these results are just a mere numerical coincidence (senseless numerology) and that indeed the values of the physical constants could actually be calculated from pure thought, rather than invoking the anthropic principle; that is, namely, based on the interplay of harmonic analysis, geometry, topology, higher dimensions, and, ultimately, number theory. The fact that the coupling constants involved the ratio of measures (volumes) may cast some light on the role of the world-sheet areas of strings, and world volumes of p-branes, as they propagate in target space–time backgrounds of diverse dimensions.

8. Conclusion

To conclude, we should add some important remarks related to string (M, F) theory and noncommutative and nonassociative geometry. Concerning string theory, we explicitly quote some of the most salient excerpts that appeared in the most recent report about the status of particle physics by Dine et al. [93]:

There are many challenges in connecting string theory to the real world, but consideration of string models has profoundly influenced ideas for particle physics models. In astroparticle physics and cosmology there is much still to explain, including the reason the cosmological constant has the value it does, the origin of cosmological density perturbations, and the nature of dark matter. String theory has had an important indirect impact on particle physics by inspiring new computational approaches to ordinary perturbation theory. String theory and supersymmetry have also had a broad impact in pure mathematics in areas ranging from algebraic geometry to number theory.

One of the most important recent developments in string theory is the AdS/CFT correspondence, or gauge/string duality. This is the startling observation that a quantum gravity theory in anti-de Sitter space is equivalent to a conformal field theory at the boundary of the space. This idea has provided a fundamental new tool for the study of strongly interacting field theories. As such it has provided a new method of studying non-perturbative QCD, has motivated new computations in lattice gauge theory, has found important applications to heavy ion physics, where it was used to predict the viscosity to entropy ratio of the quark–gluon plasma, and is now being widely applied to problems in condensed matter physics.

There has also been increasing interaction between particle theory and areas of pure mathematics, an area of research sometimes referred to as "physical mathematics". For example, there are burgeoning connections between number theory, geometry and the mathematical structure of scattering amplitudes. There has also been a resurgence of interest in the formal structure of supersymmetric gauge theories and their application to areas of mathematics, including knot theory and the structure of low-dimensional manifolds. Dualities in string theory have found a direct connection to elements of the Langlands correspondence, one of the main drivers of research in mathematics.

On the negative front, string theory gives us a *vast* number of possible vacua, 10^{500}, which is a huge "landscape" of possibilities that can be realized in a multiverse and populated by eternal inflation. Schellekens reviewed the developments in this area, focusing especially on the last decade [94]. Despite the huge number of vacua the search for realistic models can be *narrowed* down *considerably*. Thanks to very powerful algorithms in computational algebraic geometry heterotic model building on 16 specific Calabi–Yau manifolds have been constructed by [95]. These 16 special manifolds are the only ones among more than half a billion manifolds in the Kreuzer–Skarke list with a nontrivial first fundamental group. The authors [95] classified the line bundle models on these manifolds, both for $SU(5)$ and $SO(10)$ GUTs, which led to consistent supersymmetric string vacua and has three chiral families. A total of about 29 000 models is found, most of them corresponding to $SO(10)$ GUTs. These models constitute a starting point for detailed heterotic model building on Calabi–Yau manifolds in the Kreuzer–Skarke list. Therefore we should not dismiss string theory yet.

Connes noncommutative geometry [96] generalizes the concepts of ordinary geometry. As recently summarized by the authors [97]:

The geometrical setting is that of an usual manifold (space–time) described by the algebra of complex valued functions defined on it, and tensor multiplied by a finite dimensional matrix algebra. The Standard Model is described as a particular almost commutative geometry, and the corresponding Lagrangian is built from the spectrum of a generalized Dirac operator. This noncommutative geometry description of the standard model has a phenomenological predictive power and is approaching the level of maturity which enables it to confront with experiments.

The spectral action principle [98] puts gauge theories, such as the Standard Model, on the same geometrical footing as general relativity deriving a Lagrangian from a noncommutative space–time, making it possible unification with gravity. The principle is purely spectral, based on the regularization of the eigenvalues of the Dirac operator, and of its fluctuations, and the action could be derived from its fermionic counterpart via the renormalization flow in the presence of anomalies.

This noncommutative model was enhanced to include massive neutrinos and the seesaw mechanism. The most remarkable result is the possibility to predict the 126 GeV mass of the Higgs particle [98]. In the context of the spectral action and the noncommutative geometry approach to the standard model, more recently the authors [97] built a model based on a larger symmetry. The latter satisfies all the requirements to have a noncommutative manifold, and mixes gauge and spin degrees of freedom without introducing extra fermions. With this grand symmetry it is natural to have the scalar field necessary to obtain the Higgs mass in the vicinity of 126 GeV. Requiring the noncommutative space to be an almost commutative geometry (i.e., the product of manifold by a finite-dimensional internal space) gives conditions for the breaking of this grand symmetry to the Standard Model.

Model building based on nonassociative geometry has also been proposed by some authors, in particular by Wulkenhaar and Farnsworth and Boyle [99]. The theme in common with the spectral action principle in noncommutative geometry and this work is the key role played by Clifford algebras (Dirac operator). We hope to pursue further connections among them in the near future. In particular, Smith [22, 23] has suggested that within the context of algebraic quantum field theory and noncommutative geometry to get a more global theory, the local Lagrangians must be patched together. Using the eight-fold periodicity of real Clifford algebras, taking N tensor products of factors of $Cl(8)$ as $Cl(8) \otimes \ldots \otimes Cl(8) = Cl(8N)$ allows the construction of arbitrarily large real Clifford algebras as composites of lots of local $Cl(8)$ factors. By taking the completion of the union of all such $Cl(8)$-based tensor products, one gets a generalized real hyperfinite II1 von Neumann algebra factor that describes physics in terms of algebraic quantum field theory.

The appearance of von Neumann algebras in noncommutative geometry is also connected to the problem of constructing a universal gauge group that underlies the dynamical symmetries of the quantum string space–time [100]. By studying toroidal compactifications of the bosonic 26-dimensional string, the authors [100] found how certain generalized Kac–Moody symmetries, such as the monster group, arise as gauge symmetries of the resulting stringy space–time. The automorphism group of the infinite-dimensional vertex operator algebras in string theory is known to be the monster group. It is warranted to study these connections more deeply.

We finalize with a discussion of some important results based on geometric methods and applications related to nonholonomic Clifford bundles, spinors, and generalized Dirac operators. In particular, the construction of off-diagonal exact solutions for Einstein–Finsler and spinor fields in modified gravity [101]; noncommutative Dirac operators for Clifford algebroids and generalized Ricci flows; and Fedosov quantization and (non)commutative and quantum gravity [102, 103].

Novel consequences of C-space relativity can be found in ref. 104 like (*i*) the minimal length stringy uncertainty relations; (*ii*) a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the doubly special relativity framework [105]; (*iii*) an elegant *nonlinear* momentum-addition law was derived to tackle the "soccer-ball" problem in doubly special relativity; (*iv*) the generalized C-space photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary space–times, while breaking the *extended* Lorentz symmetry in C-spaces (which does *not* occur in doubly special relativity, nor in other approaches, like the presence of quantum space–time foam); and (*v*) the addition law of areal velocities and a minimal length interpretation of the length scale L, which appears in the C-space interval.

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